Discrete Inversion Formulas for the Design of Lead and Lag Discrete Compensators

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UNIVERSITÀ DEGLI STUDI
DI MODENA E REGGIO EMILIA

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Outline

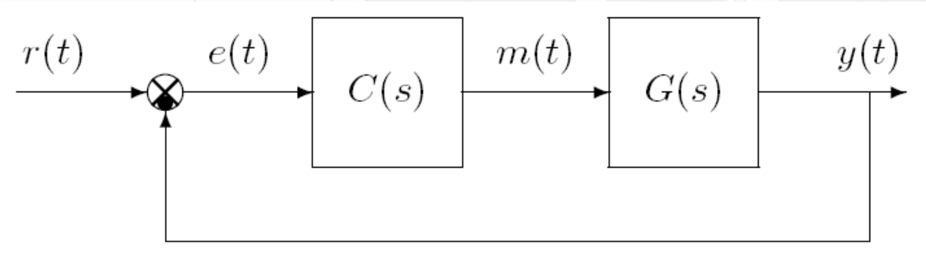
The design of first or second order compensators is a very classical topic in automatic control courses.

It can be done in a lot of different ways and usually the Bode diagrams are used. All these different ways are basically equivalent. Which is the simplest and clearest way? This is an open question!

This paper presents continuous and discrete inversion formulas which have a nice graphical interpretation on the Nyquist plane and seems to be useful for teaching.

Compensator design: the continuous time case

The considered block scheme:



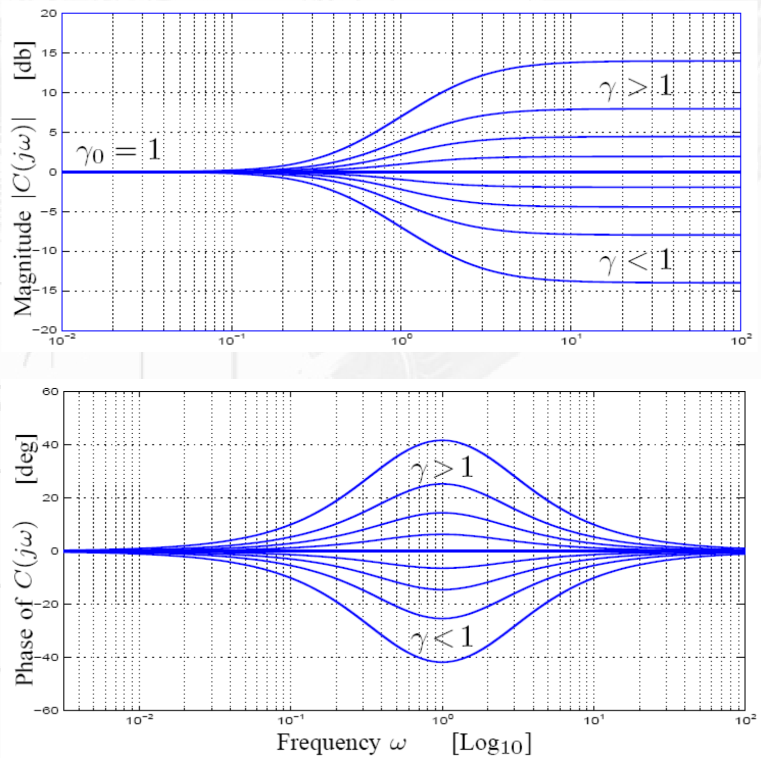
The compensator:

$$C(s) = \frac{1 + \tau_1\, s}{1 + \tau_2\, s}$$

Steady state and high frequency gains:

$$\gamma_0 = \lim_{s\to 0} C(s) = 1 \qquad \gamma = \lim_{s\to\infty} C(s) = \frac{\tau_1}{\tau_2}$$

Bode magnitude and phase plots:



Lead compensator if : $\gamma > 1$

Lag compensator if : $\gamma < 1$

Inversion Formulas - continuous time case

The design problem can often be formulated as follows. *Find the parameters of compensator C(s) such that:*

$$C(j\omega) = \frac{1 + j\,\omega\tau_1}{1 + j\,\omega\tau_2} = M\,e^{j\varphi}$$

where M *and* φ *are the magnitude and the phase desired at frequency* ω.

The problem is solved by using the following Inversion Formulas [1],[2],[3]:

$$\tau_1 = \frac{M - \cos\varphi}{\omega\sin\varphi}, \qquad \tau_2 = \frac{\cos\varphi - \frac{1}{M}}{\omega\sin\varphi}$$

Admissible domains:

$$\mathcal{D}_1 = \left\{0 \le \varphi < \frac{\pi}{2},\; M\cos\varphi \ge 1\right\} \qquad \mathcal{D}_2 = \left\{-\frac{\pi}{2} < \varphi \le 0,\; 0 < M \le \cos\varphi\right\}$$

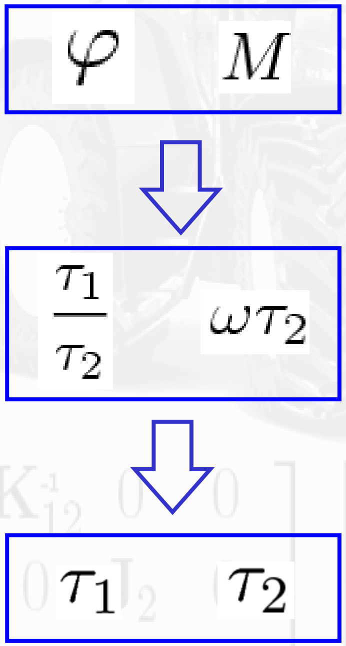
[1] The M. Policastro, G. Zonta, "Un procedimento di calcolo delle reti correttrici nella sintesi in frequenza di sistemi di controllo", University of Trieste, Internal Report no. 88, 1982.

[2] The G. Marro and R. Zanasi, "New Formulae and Graphics for Compensator Design'', 1998 IEEE International Conference On Control Applications, Trieste, Italy, 1998.

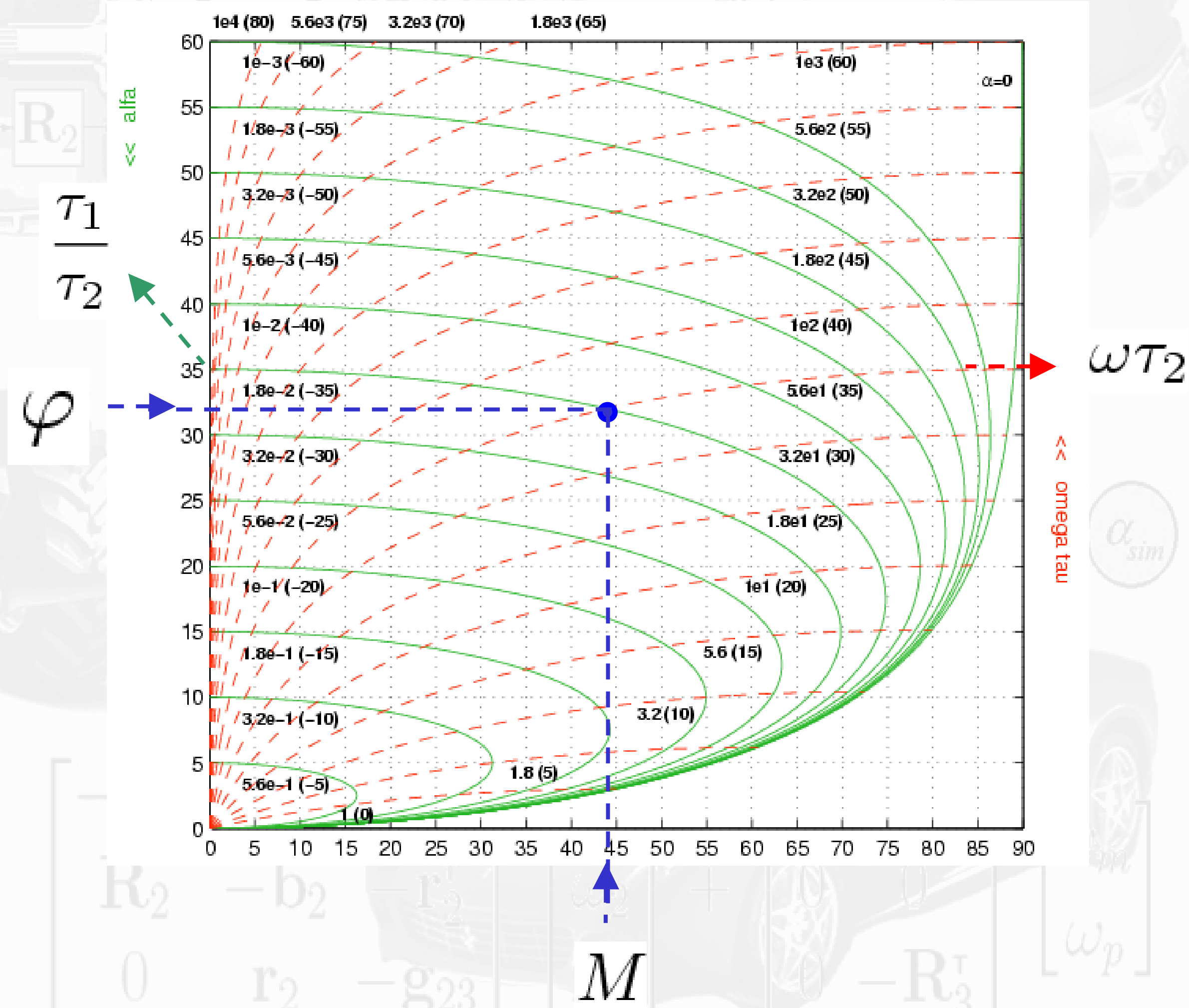
[3] The G. Marro, "Controlli Automatici'", Zanichelli, Bologna, 2004.

Without the Inversion Formulas

Without the Inversion Formulas the control problem can be solved reading the compensator parameters from lookup tables:



Lead compensator



Compensator design on the Nyquist plane

1) Chose a point B on the Nyquist Plane.

2) Red semicircle: are all the points that can be moved in B using a lead compensator.

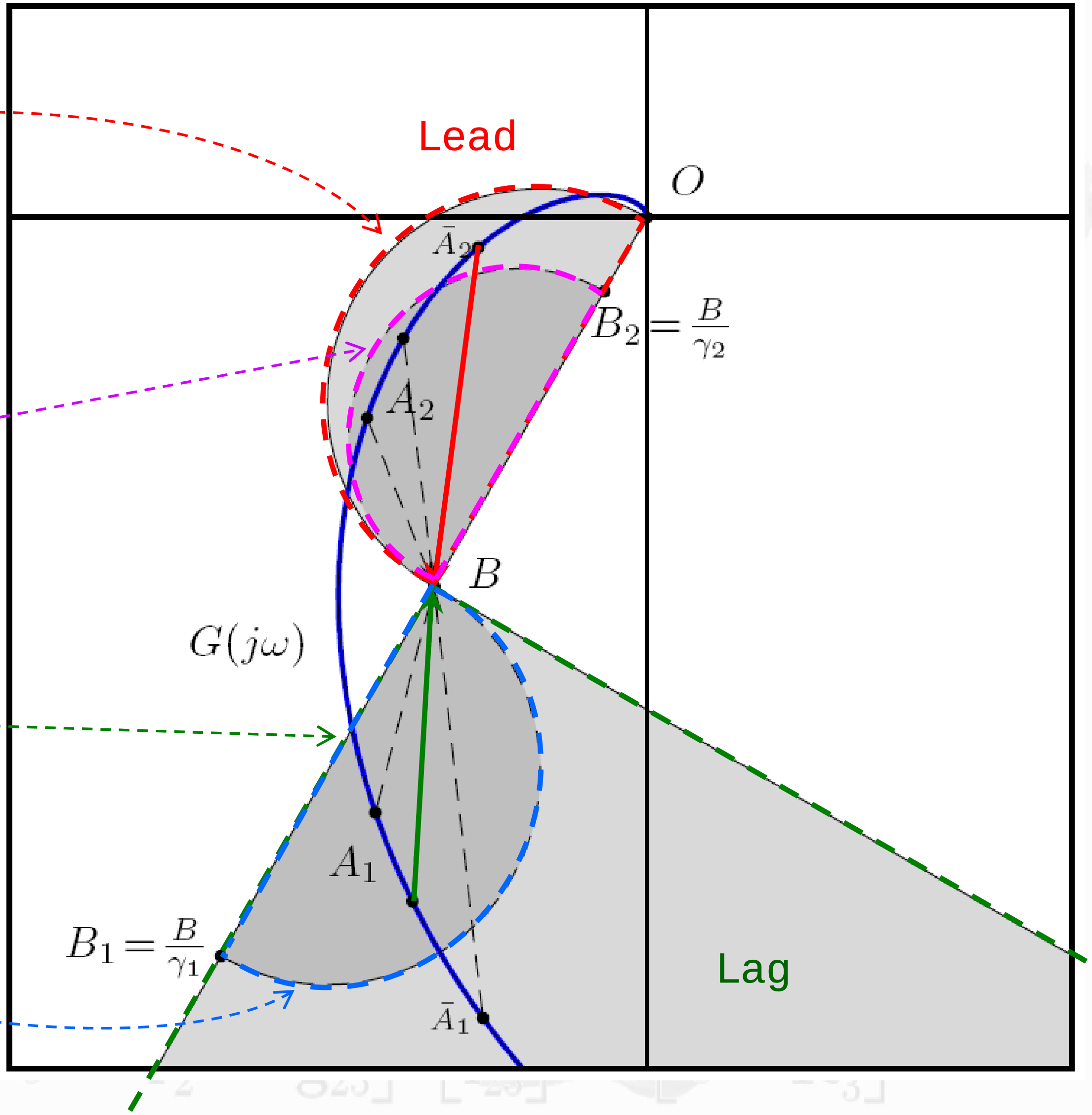
3) Lead compensators with bouded amplification:

$$\gamma < \gamma_2$$

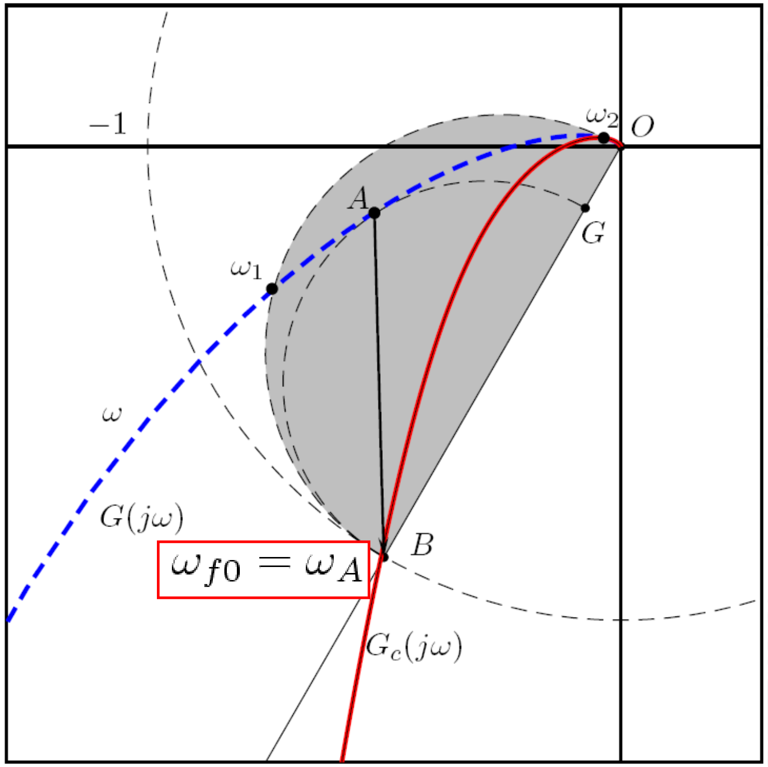
4) Green quarter of plane: are all the points that can be moved in B using a lag compensator.

5) Lag compensators with bounded attenuation:

$$\gamma_1 < \gamma$$



Lead compensator design: numerical example



The system:

$$G(s) = \frac{25}{s(s+1)(s+10)}$$

The goal:

$$M_\varphi = 60^o$$

Point B follows from the phase margin:

$$M_B = 1 \qquad \varphi_B = \pi + M_\varphi = 240^o$$

Point A is chosen within the admissible region:

$$M_A = 0.538, \qquad \varphi_A = 194.9^o, \qquad \omega_A = 2.02$$

Parameters:

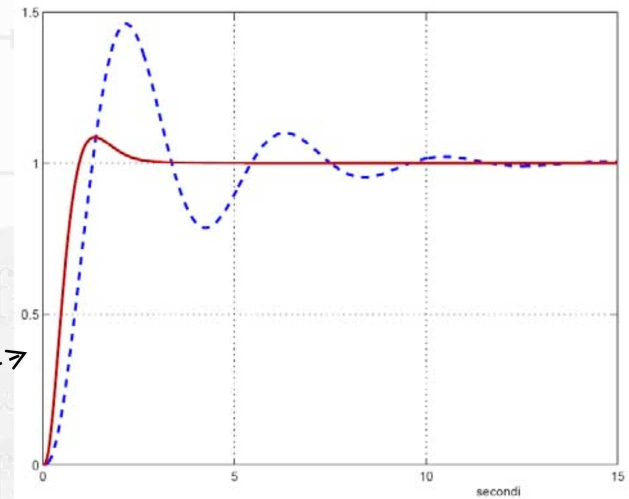
$$M = \frac{M_B}{M_A} = 1.859,$$

$$\varphi = \varphi_B - \varphi_A = 45.1^o$$

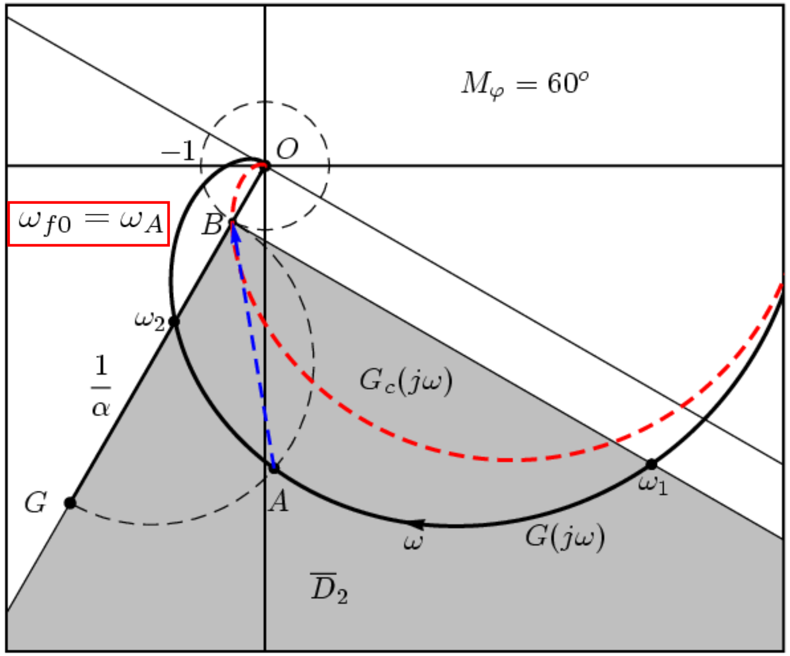
Using the Inversion Formulas one obtains the copensator:

$$C_1(s) = \frac{(1 + 0.806\, s)}{(1 + 0.117\, s)}$$

Step responces with and without the compensator.



Lag compensator design: numerical example



The system:

$$G(s) = \frac{5000}{(s+1)(s+2)(s+10)(s+30)}$$

The goal:

$$M_\varphi = 60^o$$

Point B follows from the phase margin:

$$M_B = 1 \qquad \varphi_B = \pi + M_\varphi = 240^o$$

Point A is chosen within the admissible region:

$$M_A = 4.672 \qquad \varphi_A = 271.82^o \qquad \omega_A = 1.16$$

Parameters:

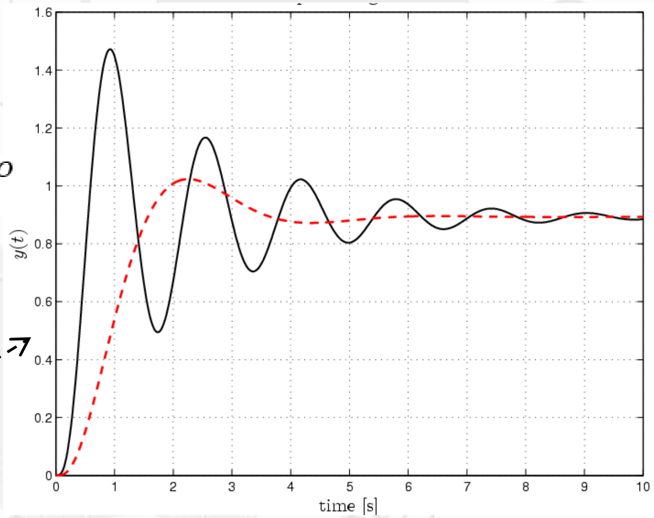
$$M = \frac{1}{M_A} = 0.214$$

$$\varphi_B - \varphi_A = -31.82^o$$

Using the Inversion Formulas one obtains the copensator:

$$C(s) = \frac{(1 + 1.04\,s)}{(1 + 6.25\,s)}$$

Step responces with and without the compensator.



Lag-Lead compensator: the continuous time case

The compensator (3 parameters):

$$C_{dg}(s) = \frac{s^2 + 2\delta_1\omega_n s + \omega_n^2}{s^2 + 2\delta_2\omega_n s + \omega_n^2}$$

The constraints:

$$\delta_1 > 0, \quad \delta_2 > 0, \quad \omega_n > 0$$

The frequency response:

$$C_{dg}(j\omega) = \frac{1 + j\,X}{1 + j\,Y}$$

where:

$$X = \delta_1 \frac{2\omega_n\omega}{\omega_n^2 - \omega^2} \qquad Y = \delta_2 \frac{2\omega_n\omega}{\omega_n^2 - \omega^2}$$

Admissible values:

$$R_3 = \{(X,\,Y) \,:\, X\,Y > 0\}$$

Inversion formulas:

$$X = \frac{M - \cos\varphi}{\sin\varphi} \qquad Y = \frac{\cos\varphi - \frac{1}{M}}{\sin\varphi}$$

Three possible design choices:

a) if you choose δ_1 :

$$\delta_2 = \delta_1 \frac{Y}{X}, \qquad \omega_n = \omega\left[\frac{\delta_1}{X} + \sqrt{\frac{\delta_1^2}{X^2} + 1}\right]$$

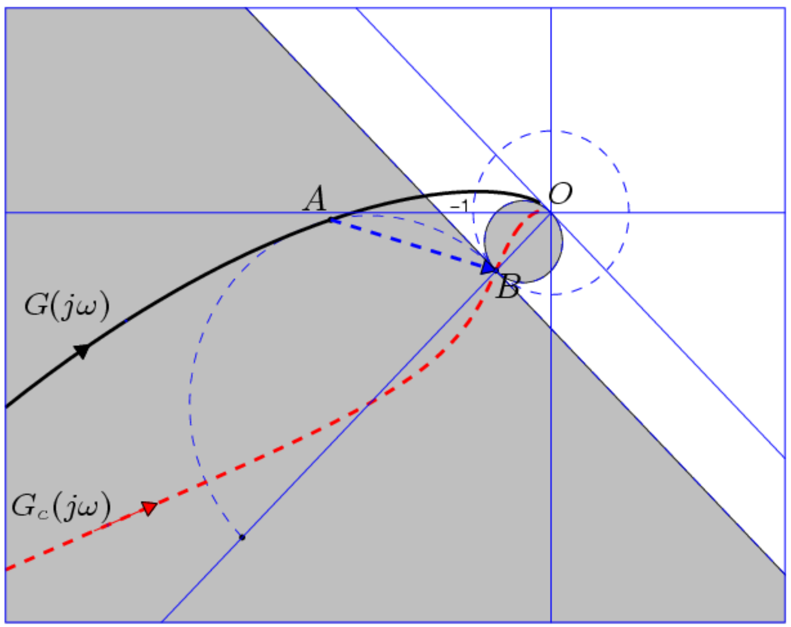
b) if you choose δ_2 :

$$\delta_1 = \delta_2 \frac{X}{Y}, \qquad \omega_n = \omega\left[\frac{\delta_2}{Y} + \sqrt{\frac{\delta_2^2}{Y^2} + 1}\right]$$

c) if you choose ω_n :

$$\delta_1 = X\frac{\omega_n^2 - \omega^2}{2\omega_n\omega}, \qquad \delta_2 = Y\frac{\omega_n^2 - \omega^2}{2\omega_n\omega}$$

Lag-Lead compensator: a numerical example



The system:

$$G(s) = \frac{280}{s(s+1)(s+10)}$$

The goal (B):

$$M_\varphi = 45^o$$

Point B follows from the phase margin:

$$M_B = 1 \qquad \varphi_B = \pi + M_\varphi = 225^o$$

Point A is chosen within the admissible region:

$$M_A = 2.83, \qquad \varphi_A = 181.7^o, \qquad \omega_A = 3.$$

Parameters:

$$M = \frac{M_B}{M_A} = 0.354$$

$$\varphi_B - \varphi_A = 43.3^o$$

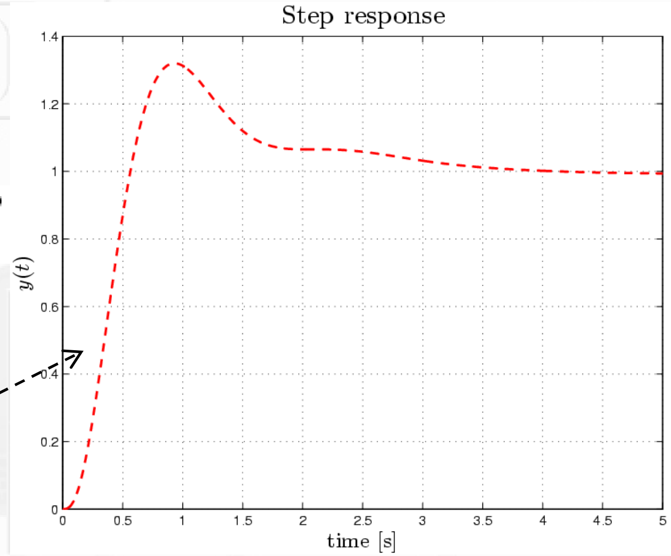
Compensator parameters:

$$\delta_1 = 0.8 \quad \delta_2 = 4.48 \quad \omega_n = 0.927$$

The lag-lead compensator:

$$C_{dg}(s) = \frac{s^2 + 1.48s + 0.859}{s^2 + 8.31s + 0.859}$$

Step responce with the compensator.

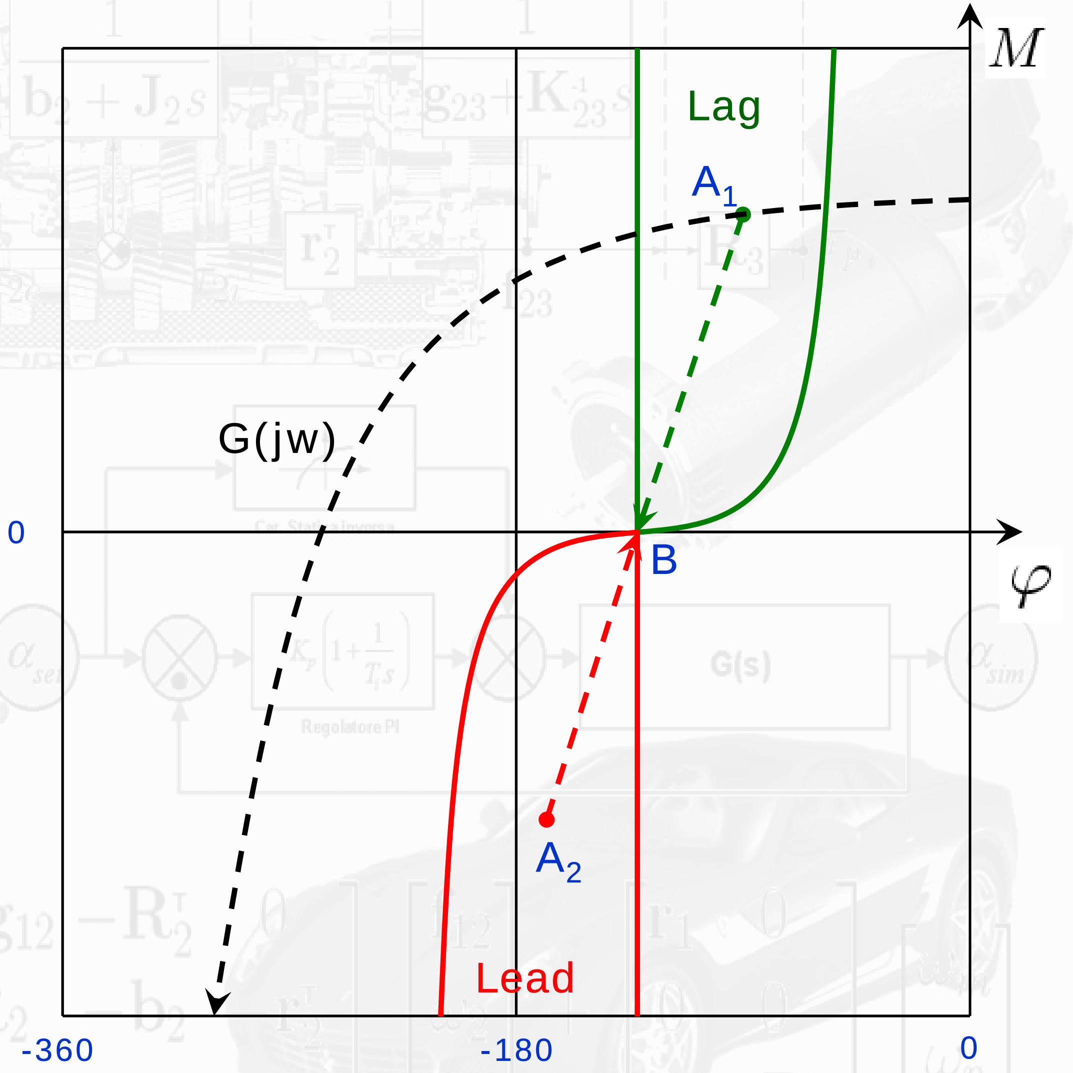


Compensator design on the Nichols plane

The Inversion Formulas can be used also on the Nichols plane.

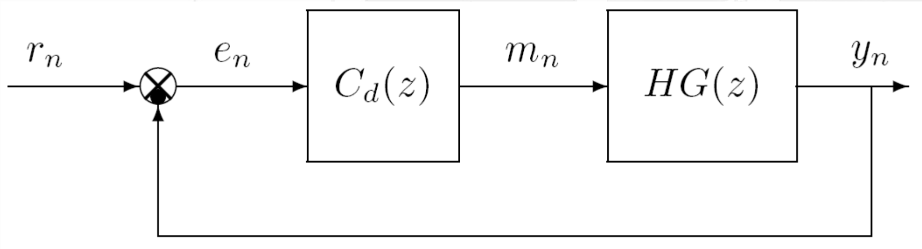
The graphical interpretation of the Lead and Lag domains is still present: they are symmetric

The Lead and Lag domains can be plotted "by hand" less precisely.

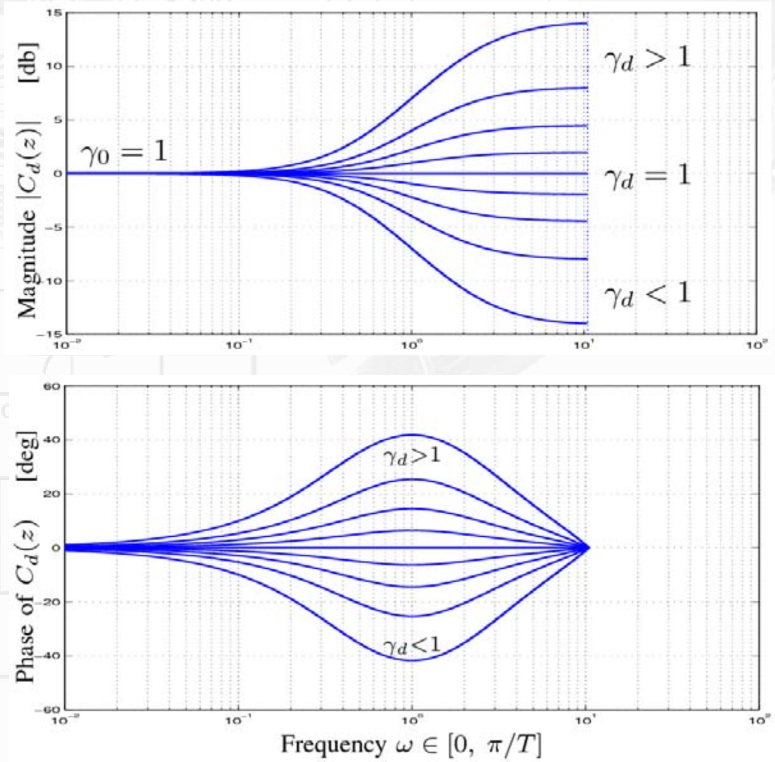


Compensator design: the discrete time case

The discrete block scheme:



Bode magnitude and phase plots:



Discrete first-order compensator:

$$C_d(z) = \frac{1 + \alpha(z-1)}{1 + \beta(z-1)}$$

$C_d(z)$ is a minimum phase system only if:

$$\alpha > 0.5 \qquad \beta > 0.5$$

Steady state and high frequency gains:

$$\gamma_0 = 1 \qquad \gamma_d = \frac{2\alpha - 1}{2\beta - 1}$$

Lead compensator if : $\alpha > \beta \quad (\gamma_d > 1)$

Lag compensator if : $\beta > \alpha \quad (\gamma_d < 1)$

Discrete Inversion Formulas

The discrete design problem can often be formulated as follows. *Find the parameters* α *and* β *of compensator* $C_d(s)$ *such that:*

$$C_d(e^{j\omega T}) = \frac{1+\alpha(e^{j\omega T}-1)}{1+\beta(e^{j\omega T}-1)} = M e^{j\varphi}$$

where M *and* φ *are the magnitude and the phase desired at frequency* ω*.*

The discrete design problem is solved by using the **Discrete Inversion Formulas**:

$$\alpha = \frac{1}{2} + \frac{M-\cos\varphi}{2\,\sin\varphi\,\tan\frac{\omega T}{2}}, \qquad \beta = \frac{1}{2} + \frac{\cos\varphi-\frac{1}{M}}{2\,\sin\varphi\,\tan\frac{\omega T}{2}}$$

$C_d(s)$ is minimum phase if and only if *$C(s)$ is minimum phase*:

$$(\alpha > 0.5,\ \beta > 0.5) \quad \Leftrightarrow \quad (\tau_1 > 0,\ \tau_2 > 0)$$

The admissible domains are equal to the ones given for the continuous time case:

$$\mathcal{D}_1 = \left\{0 \le \varphi < \frac{\pi}{2},\ M\cos\varphi \ge 1\right\} \qquad \mathcal{D}_2 = \left\{-\frac{\pi}{2} < \varphi \le 0,\ 0 < M \le \cos\varphi\right\}$$

Discrete Inversion Formulas: details

1) Auxiliary variables:

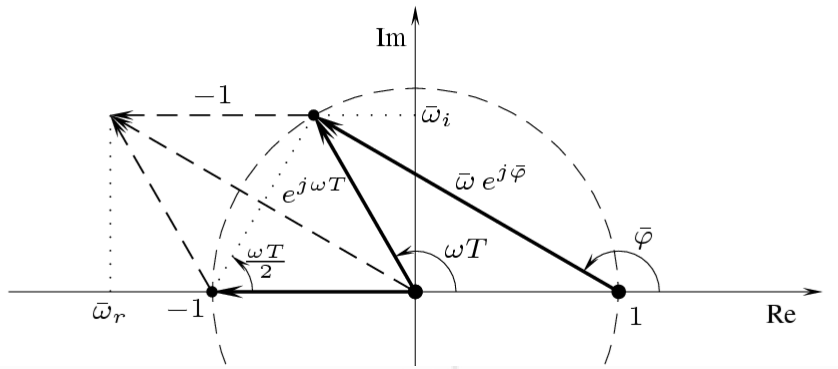
$$e^{j\omega T} - 1 = \bar{\omega}_r + j\,\bar{\omega}_i = \bar{\omega}\, e^{j\bar{\varphi}}$$

where:

$$\bar{\omega}_r = \bar{\omega}\cos\bar{\varphi}, \qquad \bar{\omega}_i = \bar{\omega}\sin\bar{\varphi}$$

and:

$$\bar{\omega} = 2\,\sin\frac{\omega T}{2} \qquad \bar{\varphi} = \frac{\pi}{2} + \frac{\omega T}{2}$$



2) Discrete design problem:

$$\frac{1 + \alpha(\bar{\omega}_r + j\bar{\omega}_i)}{1 + \beta(\bar{\omega}_r + j\bar{\omega}_i)} = M(\cos\varphi + j\sin\varphi)$$

3) The design problem can be rewritten as:

$$\begin{bmatrix} M(\bar{\omega}_r\cos\varphi - \bar{\omega}_i\sin\varphi) & -\bar{\omega}_r \\ M(\bar{\omega}_r\sin\varphi + \bar{\omega}_i\cos\varphi) & -\bar{\omega}_i \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 - M\cos\varphi \\ -M\sin\varphi \end{bmatrix}$$

4) After some manipulations:

$$\alpha = -\frac{\bar{\omega}_r\sin\varphi + \bar{\omega}_i\cos\varphi - M\bar{\omega}_i}{\bar{\omega}^2\sin\varphi}$$

$$\beta = \frac{-M\bar{\omega}_r\sin\varphi + M\bar{\omega}_i\cos\varphi - \bar{\omega}_i}{M\,\bar{\omega}^2\sin\varphi}$$

5) At the end one obtains:

$$\alpha = \frac{1}{2} + \frac{M - \cos\varphi}{2\,\sin\varphi\,\tan\frac{\omega T}{2}}$$

$$\beta = \frac{1}{2} + \frac{\cos\varphi - \frac{1}{M}}{2\,\sin\varphi\,\tan\frac{\omega T}{2}}$$

Discrete Inversion Formulas: remarks

The discrete inversion formulas can also be rewritten in the following form:

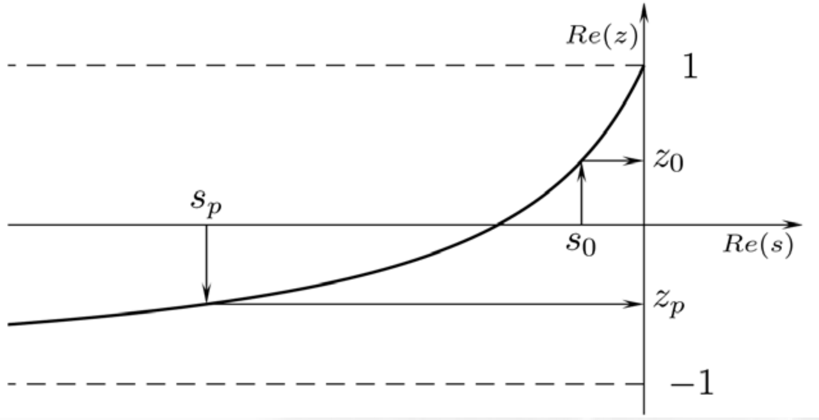
$$\alpha = \frac{1}{2} + \frac{\omega\,\tau_1}{2\,\tan\frac{\omega T}{2}}, \qquad \beta = \frac{1}{2} + \frac{\omega\,\tau_2}{2\,\tan\frac{\omega T}{2}}$$

where τ_1 and τ_2 are the parameters given by continuous time inversion formulas.

Poles and zeros of the continuous and discrete inversion formulas:

$$s_p = -\frac{1}{\tau_2} \qquad s_0 = -\frac{1}{\tau_1}$$

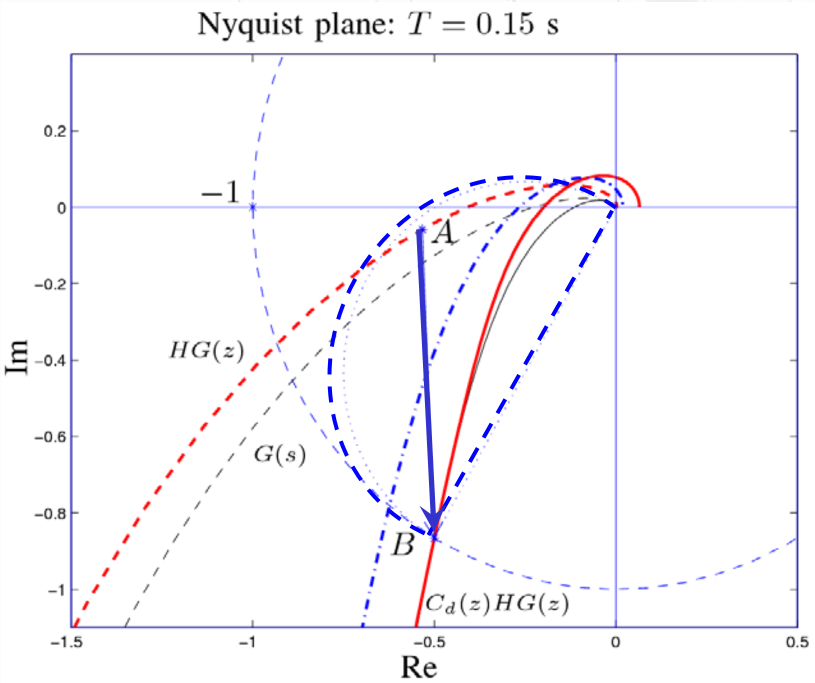
$$z_p = 1 - \frac{1}{\beta} \qquad z_0 = 1 - \frac{1}{\alpha}$$



C(s) is transformed in *C_d(s)* by using the "*bilinear transformation with prewarping*":

$$C(s) = \frac{1+\tau_1\,s}{1+\tau_2\,s} \quad + \quad s = \frac{\omega}{\tan\frac{\omega T}{2}}\left[\frac{z-1}{z+1}\right] \quad = \quad C_d(z) = \frac{1+\alpha(z-1)}{1+\beta(z-1)}$$

Discrete inversion formulas: numerical example



The system:

$$G(s) = \frac{25}{s(s+1)(s+10)}$$

The goal (B):

$$M_\varphi = 60^o$$

The discrete system:

$$HG(z) = \frac{(9.657z^2 + 26.66z + 4.259)10^{-3}}{z^3 - 2.084z^2 + 1.276z - 0.192}$$

Point A is chosen within the admissible region:

$$M_A = 0.5361, \qquad \varphi_A = 186.2^o, \qquad \omega_A = 2.02.$$

Parameters:

$$M = \frac{M_B}{M_A} = 1.865$$

$$\varphi = \varphi_B - \varphi_A = 53.76^o$$

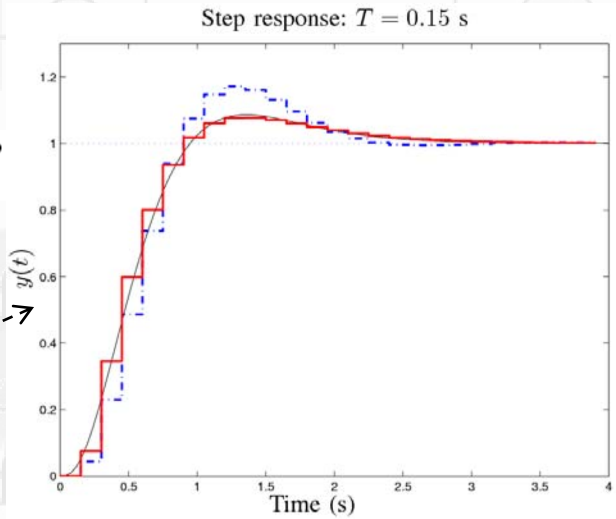
With discrete inversion formulas:

$$C_d(z) = \frac{1 + 5.673(z-1)}{1 + 0.723(z-1)}$$

With bilinar transformaion:

$$C_b(z) = \frac{1.762\,z - 1.462}{0.384\,z - 0.084}$$

Step responces with continuous, discrete and bilinear compensators.



Discrete inversion formulas: remarks

Frequency parameters to move A -> B :

$$M = \frac{M_B}{M_A} = 1.865 \qquad \varphi = \varphi_B - \varphi_A = 53.76^o \qquad \omega = 2.02$$

Using the discrete inversion formulas:

$$\alpha = \frac{1}{2} + \frac{M - \cos\varphi}{2\,\sin\varphi\,\tan\frac{\omega T}{2}}$$

$$\beta = \frac{1}{2} + \frac{\cos\varphi - \frac{1}{M}}{2\,\sin\varphi\,\tan\frac{\omega T}{2}}$$

one obtains the discrete compensator:

$$C_d(z) = \frac{1+\alpha(z-1)}{1+\beta(z-1)} = \frac{1+5.673(z-1)}{1+0.723(z-1)}$$

Using the continuous inversion formulas:

$$\tau_1 = \frac{M - \cos\varphi}{\omega\sin\varphi}, \qquad \tau_2 = \frac{\cos\varphi - \frac{1}{M}}{\omega\sin\varphi}$$

one obtains:

$$C(s) = \frac{1+\tau_1 s}{1+\tau_2 s} = \frac{1+0.782\,s}{1+0.0337\,s}$$

Then applying the "*bilinear transformation with prewarping*":

$$s = \frac{\omega}{\tan\frac{\omega T}{2}}\left[\frac{z-1}{z+1}\right]$$

one obtains the same discrete compensator !!!

Conclusions

1) Inversion Formulas – continuous case

- Easy to use (Lead, Lag and Lead/Lag compensators)
- Simple graphical representations on Nyquist plane
- Easy understandable by undergraduate students

2) Inversion Formulas – discrete case

- Similar to the continuous time case
- Same domains and same graphical representation on the Nyquist plane
- Useful for teaching and for discrete control design.