SIMULATION OF VARIABLE DYNAMIC DIMENSION SYSTEMS: THE CLUTCH EXAMPLE

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Abstract

Due to the presence of strong non-linearities (i.e. coulomb friction) some systems change their dynamic dimension while functioning. Such systems can be found in many application fields, such as in automotive and robotics. This paper proposes a simulation model for this type of systems that we will call as "Variable Dynamic Dimension Systems" (VDDS). Particularly, a VDDS system composed by n masses which slip and interact together by means of the coulomb friction is analyzed. A congruent state space transformation is used to obtain a simple and effective simulation model for the system. The performances of the model are tested through simulation experiments applied to an interesting automotive application: a clutch with torsional damper-spring.

1 Introduction

The typical time-varying nonlinear systems are usually described with the equation $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$ where $\mathbf{x} = (x_1, x_2, ..., x_n)$ is the state vector, $\mathbf{u} = (u_1, u_2, ..., u_m)$ the input vector and t the time. The dynamic dimension of these systems is n and usually it is constant. However when strong non-linearities (i.e. coulomb friction) affect the system dynamics, it can happen that the system behaves as a lower order system. We will call such systems as "Variable Dynamic Dimension Systems" (VDDS). A typical example of such systems is given by a clutch (see Fig. 1). When the clutch is slipping, the two inertias J_1 and J_2 move independently (one respect to the other) under the action of the torques C_1 and C_2, and only the coulomb friction τ_{12} is exchanged between them. Otherwise, when the clutch is locked, the two inertias rotate together. In this working condition the order of the model is equal to 1, and it is easily described by a first order model where the torque $C_1 - C_2$ acts on the inertia $J_1 + J_2$. In many application fields, such as in mechanics and in robotics, "variable dynamic dimension systems" have to be handled. The simulation of such systems is not easy due to the fact that the model changes its order. This leads to models that are or exact but huge and very complex, or simple but approximated. In this paper a new model for the exact simulation of "variable dynamic dimension systems" is proposed. The paper is organized as follows. In Section 2 the simulation problem is stated and the typical schemes used to simulate such problem are presented. The proposed simulation model is discussed in Section 3. A detailed example explaining the features of the proposed model is presented and simulated in Section 4. Finally, an interesting application to an automotive system, a clutch with torsional damper-spring, is described and simulated in Section 5.

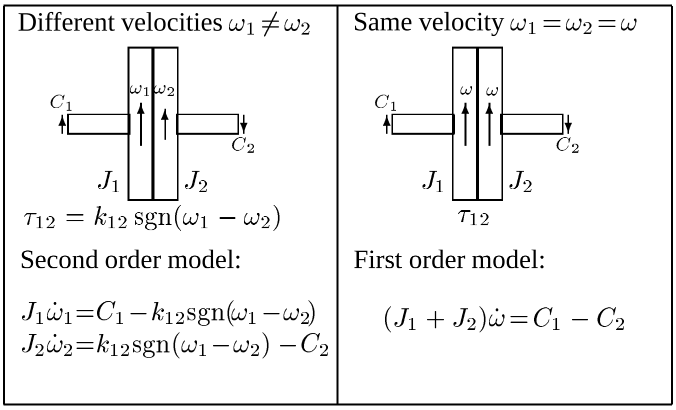

Figure 1: Two different dynamic models for the clutch system.

2 Problem statement

In this paper, systems having the framework shown in Fig. 2 are considered. In the general case, the n masses (or inertias) m_1, m_2,... ,m_n move (or rotate) under the action of the n external forces (or torques) F_1, F_2,..., F_n. Let x_1, x_2,..., x_n be the positions (angles) of the masses (inertias). Due to the presence of the coulomb friction, the motion of a single mass depends also on the motions of the neighborhood masses: $\tau_{i,i+1}$ is the coulomb friction between the masses m_i and m_{i+1}. Let $v_1 = \dot{x}_1$, ..., $v_n = \dot{x}_n$ denote the linear velocities of the n masses. The system dynamics is given by the following differential equations:

$$m_i \dot{v}_i = F_i + \tau_{i-1,i} - \tau_{i,i+1} \quad \text{for} \quad i = 1, ..., n \tag{1}$$

where $\tau_{i,i+1}$ is the amplitude of the coulomb friction given by:

$$\tau_{i,i+1} = \begin{cases} k_{i,i+1}\,\mathrm{sgn}(v_i - v_{i+1}), & \text{if} \quad v_i \neq v_{i+1} \\ k_{i,i+1}\,\mathrm{sat}\left(\dfrac{(F_i + \tau_{i-1,i})m_{i+1} - (F_{i+1} - \tau_{i+1,i+2})m_i}{k_{i,i+1}(m_i + m_{i+1})} \right) & \text{if} \quad v_i = v_{i+1} \end{cases} \tag{2}$$

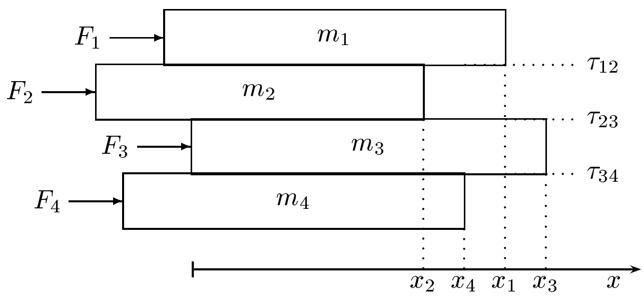

Figure 2: Example of a variable dynamic dimension system

where $k_{i,i+1}$ is the coulomb friction coefficient (that is the maximum absolute value of the coulomb friction). When $v_i = v_{i+1}$, the coulomb friction is equal to the force $\tau_{i,i+1}$ that keeps the relative velocity $v_i - v_{i+1}$ equal to zero. The force $\tau_{i,i+1}$ has a limited amplitude: $k_{i,i+1}$. When this amplitude is exceeded, the two masses m_i and m_{i+1} start to move at different velocities. Otherwise, when $|\tau_{i,i+1}| < k_{i,i+1}$ the value of the force $\tau_{i,i+1}$ is the following, see eq. (2),:

$$\tau_{i,i+1} = \frac{(F_i + \tau_{i-1,i})m_{i+1} - (F_{i+1} - \tau_{i+1,i+2})m_i}{m_i + m_{i+1}}$$

In fact, substituting this result in equation (1) one obtains:

$$\begin{aligned}(m_i + m_{i+1})\dot{v}_i &= (F_i + \tau_{i-1,i}) + (F_{i+1} - \tau_{i+1,i+2})\\ (m_i + m_{i+1})\dot{v}_{i+1} &= (F_i + \tau_{i-1,i}) + (F_{i+1} - \tau_{i+1,i+2})\end{aligned}$$

that is, the two masses m_i and m_{i+1} move together (as a single mass $m_i + m_{i+1}$) at the same velocity $\dot{v}_i = \dot{v}_{i+1}$ under the action of the resulting force $(F_i + \tau_{i-1,i}) + (F_{i+1} - \tau_{i+1,i+2})$. In this condition, the dynamic dimension of the system reduces to $n - 1$. If the coulomb friction between the masses is able to keep l relative velocities to zero, the dynamic dimension of the system reduces to $n - l$. The coulomb friction between the masses influences and is influenced by the dynamics of the whole system. Due to the particular form of the coulomb friction given in (2), the exact simulation of the "variable dynamic dimension systems" is particularly difficult.

2.1 Multi-subsystem simulation model

The typical scheme used to simulate the "variable dynamic dimension systems" is strictly based on the definition of the coulomb friction given in (2). When the system is composed by n masses, there are $n - 1$ relative velocities that have to be checked to choose which one of the relations of equation (2) has to be used, consequently there are $m = 2^{n-1}$ different possible configurations (see [1]). The block scheme used in this case in shown in Fig. 3: at each instant the "selection module" chooses which model has to be used. With this type of simulation model, the following problems arise:

a) The number m of different models increases exponentially with the number n of masses: $m = 2^{n-1}$.

b) When the system switches from a model to another, the updating of the initial conditions of the new model is required, that is, the n state variables of the system must be converted

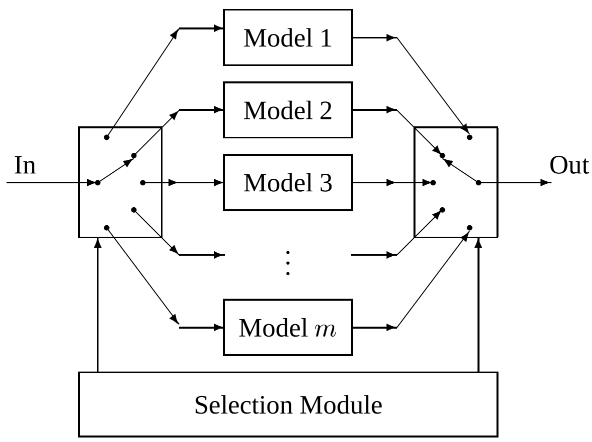

Figure 3: Framework of the multi-subsystem simulation model

from the old model to the new one.

c) Inputs and outputs of the simulator have to be properly connected to the new model (see Fig. 3).

d) At each instant of the simulation time, the "selection module" has to choose which one, among all the possible models, is the correct model to be use at that time.

For all these reasons, the multi-subsystem simulation model shown in Fig. 3 is generally huge and very complex to be used when the number of masses is greater than 3.

2.2 Sliding mode simulation model

To cope with the all these problems, a different approach, mainly based on the sliding mode theory, is commonly used. The sliding mode simulation model of the considered system can be obtained from (1)-(2) by replacing the second part of equation (2) with equation $\tau_{i,i+1} = k_{i,i+1}\,\mathrm{sgn}(v_i - v_{i+1})$, which is exactly equal to the former part of equation (2). In this case the coulomb friction is seen as a sliding input variable for the system. When the previous equation is used and the relative velocities are not zero, the dynamic behaviour of the system remains the same as when equation (2) is used. On the contrary, when the relative velocity $v_i - v_{i+1}$ becomes zero, the corresponding coulomb friction $\tau_{i,i+1}$ starts switching at infinite frequency between the two values $\pm k_{i,i+1}$ trying to keep to zero the relative velocity $v_i - v_{i+1}$ (see Fig. 2). The continuous-time equivalent value of the sliding variable $\tau_{i,i+1}$ can be easily computed as shown in [6], and the corresponding dynamic behaviour is fully equivalent to the second relation of equation (2). In this case, just one model is used to simulate the behaviour of the whole system in every functional condition. Unfortunately, this sliding model can not be "exactly" simulated. In fact, all the simulators can not handle an ideal infinite switching frequency. Therefore a "finite" switching frequency is really applied to the model and this introduces a strong approximation, as shown later in Section 4. Moreover, to improve the simulation accuracy a very small integration step is necessary and therefore simulation time increases dramatically. To avoid the very high switching frequency of the sliding variables

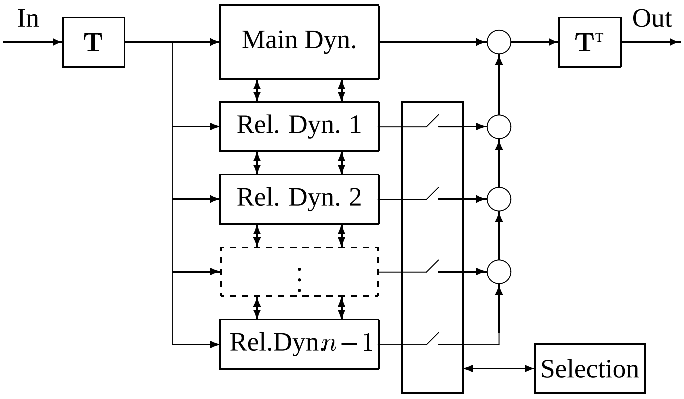

Figure 4: *Variable dynamic dimension systems*: block scheme.

and to reduce the simulation time, the sign function "$\mathrm{sgn}()$" is sometimes substituted by the saturation function "$\mathrm{sat}()$", but this solution is even more approximated than the previous.

3 Proposed simulation model for variable dynamic dimension systems

The complexity of the variable dynamic dimension systems lies in the definition of the coulomb friction given by equation (2). The basic idea of the proposed simulation model, shown in Fig. 4, is the introduction of a proper congruent state space transformation $\mathbf{T}$ that allows to see the system dynamics from a new perspective that decouples the main system dynamics from all the other relative dynamics. The transformation $\mathbf{T}$ shows that the coulomb friction $\tau_{i,i+1}$ acts only on the relative dynamics given by the relative velocity $v_i - v_{i+1}$. Consequently, $\mathbf{T}$ decouples the dynamics of the original system into n independent dynamics. The Main Dynamics shown in Fig. 4 describes the lower dimensional dynamics of the whole system, namely the dynamics of the system when all the bodies move together. The remainder $n-1$ submodels describe the $n-1$ Relative Dynamics among the masses. The Main Dynamics and the $n-1$ Relative Dynamics are simply obtained by applying the state space transformation $\mathbf{T}$. In the transformed space, the task of the Selection Module is just to add or remove the relative dynamics when the dynamic dimension of the system increases or decreases: the i-th Relative Dynamics is added when the mass m_i slides on mass m_{i+1}, and it is removed when m_i and m_{i+1} move together. The use of the transformation $\mathbf{T}$ is very simple and leads to a simulation model much easier than the solutions shown in the previous section. Let us now again consider the system shown in Fig. 2. Four masses m_i, $i \in \{1,2,3,4\}$, move in the x direction subjected to forces F_i. The masses interact by means of the coulomb friction which, in this case, is completely described by the friction coefficients k_{12}, k_{23} and k_{34}. The positions of the four masses are given by the variables x_i. Let us denote with v_1, v_2, v_3 and v_4 the velocities of the four masses: $v_1 = \dot{x}_1$, $v_2 = \dot{x}_2$, $v_3 = \dot{x}_3$, $v_4 = \dot{x}_4$. The differential equations describing the dynamics of the considered system in slipping conditions are:

$$\begin{cases} m_1\,\dot{v}_1 = F_1 - k_{12}\,\mathrm{sgn}(v_1-v_2) \\ m_2\,\dot{v}_2 = F_2 + k_{12}\,\mathrm{sgn}(v_1-v_2) - k_{23}\,\mathrm{sgn}(v_2-v_3) \\ m_3\,\dot{v}_3 = F_3 + k_{23}\,\mathrm{sgn}(v_2-v_3) - k_{34}\,\mathrm{sgn}(v_3-v_4) \\ m_4\,\dot{v}_4 = F_4 + k_{34}\,\mathrm{sgn}(v_3-v_4) \end{cases} \qquad (3)$$

Let us now define the following vectors and matrices:

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & m_3 & \\ & & & m_4 \end{bmatrix}, \; \mathbf{D} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \; \mathbf{K} = \begin{bmatrix} k_{12} & & \\ & k_{23} & \\ & & k_{34} \end{bmatrix}$$

System (3) can be rewritten in matrix form as follows:

$$\mathbf{M}\,\dot{\mathbf{v}} = \mathbf{F} - \mathbf{D}^{\mathrm{T}}\mathbf{K}\,\mathrm{sgn}(\mathbf{D}\mathbf{v}) \quad \leftrightarrow \quad \mathbf{M}\,\dot{\mathbf{v}} = \mathbf{F} - \mathbf{E} \qquad (4)$$

Vectors $\mathbf{w} = \mathbf{D}\mathbf{v}$ and $\mathbf{E} = \mathbf{D}^{\mathrm{T}}\mathbf{K}\,\mathrm{sgn}(\mathbf{D}\mathbf{v})$ are:

$$\mathbf{w} = \begin{bmatrix} v_1 - v_2 \\ v_2 - v_3 \\ v_3 - v_4 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} k_{12}\,\mathrm{sgn}(v_1-v_2) \\ k_{23}\,\mathrm{sgn}(v_2-v_3) - k_{12}\,\mathrm{sgn}(v_1-v_2) \\ k_{34}\,\mathrm{sgn}(v_3-v_4) - k_{23}\,\mathrm{sgn}(v_2-v_3) \\ -k_{34}\,\mathrm{sgn}(v_3-v_4) \end{bmatrix}$$

System (3) can be simplified if the following state space congruent transformation is considered:

$$\mathbf{v} = \mathbf{T}\,\mathbf{z}, \; \mathbf{T} = [\mathbf{T}_1 | \mathbf{T}_2] = \left[\begin{array}{c|ccc} 1 & \frac{m_2+m_3+m_4}{\Delta} & \frac{m_3+m_4}{\Delta} & \frac{m_4}{\Delta} \\ 1 & -\frac{m_1}{\Delta} & \frac{m_3+m_4}{\Delta} & \frac{m_4}{\Delta} \\ 1 & -\frac{m_1}{\Delta} & -\frac{m_1+m_2}{\Delta} & \frac{m_4}{\Delta} \\ 1 & -\frac{m_1}{\Delta} & -\frac{m_1+m_2}{\Delta} & -\frac{m_1+m_2+m_3}{\Delta} \end{array}\right]$$

where $\Delta = m_1+m_2+m_3+m_4$. The new state vector $\mathbf{z}$ has the following physical meaning:

$$\mathbf{z} = \mathbf{T}^{-1}\mathbf{v} \;\; \leftrightarrow \;\; \mathbf{z} = \begin{bmatrix} \frac{1}{\Delta}(m_1 v_1 + m_2 v_2 + m_3 v_3 + m_4 v_4) \\ v_1 - v_2 \\ v_2 - v_3 \\ v_3 - v_4 \end{bmatrix}$$

In this case, the *Main Dynamics* is described by the transformed velocity z_1 (the mean velocity weighted by the masses), while the *Relative Dynamics* are described by the other three relative velocities z_2, z_3 and z_4. By using $\mathbf{v} = \mathbf{T}\mathbf{z}$, system (4) transforms as $\mathbf{M}_T\,\dot{\mathbf{z}} = \mathbf{F}_T - \mathbf{E}_T$:

$$\underbrace{\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}}_{\mathbf{M}_T}\,\dot{\mathbf{z}} = \underbrace{\mathbf{T}^{\mathrm{T}}\mathbf{F}}_{\mathbf{F}_T} - \underbrace{(\mathbf{D}\mathbf{T})^{\mathrm{T}}\mathbf{K}\,\mathrm{sgn}(\overbrace{\mathbf{D}\mathbf{T}\mathbf{z}}^{\mathbf{w}})}_{\mathbf{E}_T} \qquad (5)$$

The new vectors $\mathbf{w} = \mathbf{D}\,\mathbf{T}\,\mathbf{z}$ and $\mathbf{E}_T$ have now a very simple structure:

$$\mathbf{w} = \begin{bmatrix} z_2 \\ z_3 \\ z_4 \end{bmatrix}, \qquad \mathbf{E}_T = \begin{bmatrix} 0 \\ k_{12}\,\mathrm{sgn} z_2 \\ k_{23}\,\mathrm{sgn} z_3 \\ k_{34}\,\mathrm{sgn} z_4 \end{bmatrix}$$

The benefit of using transformation $\mathbf{T}$ is that now matrices $\mathbf{M}_T = \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}$ and $\mathbf{M}_T^{-1}$ are block diagonal matrices.

$$\mathbf{M}_T^{-1} = \left[\begin{array}{c|ccc} \frac{1}{\Delta} & 0 & 0 & 0 \\ \hline 0 & \frac{m_1+m_2}{m_1 m_2} & -\frac{1}{m_2} & 0 \\ 0 & -\frac{1}{m_2} & \frac{m_2+m_3}{m_2 m_3} & -\frac{1}{m_3} \\ 0 & 0 & -\frac{1}{m_3} & \frac{m_3+m_4}{m_3 m_4} \end{array}\right] = \left[\begin{array}{c|c} M_{T1}^{-1} & 0 \\ \hline 0 & \mathbf{M}_{T2}^{-1} \end{array}\right]$$

Let vectors $\mathbf{z}$, $\mathbf{F}_T$ and $\mathbf{E}_T$ be partitioned as follows:

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \hline z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ \mathbf{z}_2 \end{bmatrix}, \mathbf{F}_T = \begin{bmatrix} F_{T1} \\ \hline F_{T2} \\ F_{T3} \\ F_{T4} \end{bmatrix} = \begin{bmatrix} F_{T1} \\ \mathbf{F}_{T2} \end{bmatrix}, \mathbf{E}_T = \begin{bmatrix} 0 \\ \hline E_2 \\ E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{E}_{T2} \end{bmatrix}$$

Since $\mathbf{M}_T$ is a block diagonal matrix, system (5) can be rewritten as:

$$\begin{cases} M_{T1}\dot{z}_1 = F_{T1} \\ \mathbf{M}_{T2}\dot{\mathbf{z}}_2 = (\mathbf{F}_{T2} - \mathbf{E}_{T2}) \end{cases} \leftrightarrow \begin{cases} \dot{z}_1 = M_{T1}^{-1} F_{T1} \\ \dot{\mathbf{z}}_2 = \mathbf{M}_{T2}^{-1}(\mathbf{F}_{T2} - \mathbf{E}_{T2}) \end{cases} \quad (6)$$

that is:

$$\begin{cases} \dot{z}_1 = \frac{F_{T1}}{\Delta} \\ \begin{bmatrix} \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \frac{m_1+m_2}{m_1 m_2} & -\frac{1}{m_2} & 0 \\ -\frac{1}{m_2} & \frac{m_2+m_3}{m_2 m_3} & -\frac{1}{m_3} \\ 0 & -\frac{1}{m_3} & \frac{m_3+m_4}{m_3 m_4} \end{bmatrix} \begin{bmatrix} F_{T2} - k_{12}\,\mathrm{sgn} z_2 \\ F_{T3} - k_{23}\,\mathrm{sgn} z_3 \\ F_{T4} - k_{34}\,\mathrm{sgn} z_4 \end{bmatrix} \end{cases} \quad (7)$$

The state space transformation $\mathbf{v} = \mathbf{T}\,\mathbf{z}$ decouples the original system in two independent parallel systems: variable z_1 is not influenced by variables z_2, z_3 and z_4 and viceversa. The second part of system (7) describes the relative dynamics and can be interpreted as a three dimensional *Variable Structure System*. When $z_j = 0$, if relation $|\tau_{j-1,j}| < k_{j-1,j}$ is satisfied, a *sliding* mode arises in the system: $\tau_{j-1,j}$ is the *equivalent control* associated with the switching term $k_{j-1,j}\,\mathrm{sgn}(z_j)$, that is the time mean value of the term $k_{j-1,j}\,\mathrm{sgn}(z_j)$, and it is equal to the force that at each instant the two masses m_{j-1} and m_j exchange due to the presence of the coulomb friction. When all the conditions $|\tau_{j-1,j}| < k_{j-1,j}$ are satisfied, in a finite time the system (7) converges towards the sliding manifold $z_2 = 0$, $z_3 = 0$, $z_4 = 0$. When one of the sliding surfaces $z_i = 0$ is reached, a *sliding mode* can arise in the system, that is the term $k_{j-1,j}\,\mathrm{sgn}(z_j)$ can start to switch at infinite frequency keeping the variable z_j at zero. In this condition the dynamic dimension of the model decreases of one unit. Since matrix $\mathbf{M}_{T2}^{-1}$ is not diagonal, when one of the sliding variables is equal to zero, its equivalent control influences immediately all the other relative dynamics. To correctly simulate the system (7), the following simulation algorithm has been designed:

loop $(z_j = 0)_{j=2,3,4}$

$\quad \tau_{j-1,j} = M_j^{-1}\,[M_i(F_{i-1,i} - k_{i-1,i}\,\mathrm{sgn}(z_i))] + F_{j-1,j}$

$\quad$ if $|\tau_{j-1,j}| < k_{j-1,j}$

$\qquad (\dot{z}_j)_{k+1} = 0$

$\quad$ else

$\qquad (\dot{z}_j)_{k+1} = M_i^{-1}\,(F_{i-1,i} - k_{i-1,i}\,\mathrm{sgn}(z_i))$

$\quad$ end if

end loop

where M_j is the matrix obtained by selecting the rows and the columns of matrix $\mathbf{M}_{T2}^{-1}$ corresponding to the sliding variables $z_j = 0$; M_i is the sub-matrix of matrix $\mathbf{M}_{T2}^{-1}$ obtained by selecting the rows corresponding to the sliding variables $z_j = 0$ and the columns corresponding the other variables $z_j \neq 0$. The main functions of this algorithm are the following:

a) it checks if the relative velocities z_j are equal to zero;
b) it computes the possible equivalent control $\tau_{j-1,j}$ related to the transformed variables z_j;
c) it verifies the sliding mode conditions $|\tau_{j-1,j}| < k_{j-1,j}$ on variables z_j;
d) it keeps at zero all the variables that satisfy the sliding mode conditions; for these variables it imposes $(\dot{z}_j)_{k+1} = 0$;
e) it computes the proper input $(\dot{z}_j)_{k+1}$ for all the variables not satisfying the sliding mode conditions.

Note: the proposed algorithm works correctly also when the parameters $k_{i-1,i}$ are time-varying. For understanding the computational method used for solving system (7), let us consider the following two cases:

I) The case when all the three variables are equal to zero: $z_2 = z_3 = z_4 = 0$. In this condition a *sliding mode* arises in the system iff:

$$|F_{T2}| < k_{12}, \qquad |F_{T3}| < k_{23}, \qquad |F_{T4}| < k_{34} \quad (8)$$

The $\mathrm{sgn}(z_i)$ functions start switching between the two values $\pm k_{i-1,i}$ at infinity frequency with an average value that, at each instant, is equal to the external transformed force F_{Ti}. If conditions (8) are satisfied, variables z_2, z_3, and z_4 are kept to zero.

II) The case when $z_2 \neq 0$ and $z_3 = z_4 = 0$. Solving system (7) with respect to the *equivalent controls* $\tau_{23} = k_{23}\,\mathrm{sgn} z_3$ and $\tau_{34} = k_{34}\,\mathrm{sgn} z_4$ one obtains:

$$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{m_2} & \frac{m_2+m_3}{m_2 m_3} & -\frac{1}{m_3} \\ 0 & -\frac{1}{m_3} & \frac{m_3+m_4}{m_3 m_4} \end{bmatrix} \begin{bmatrix} F_{T2} - k_{12}\,\mathrm{sgn} z_2 \\ F_{T3} - \tau_{23} \\ F_{T4} - \tau_{34} \end{bmatrix} \right.$$

that is:

$$\left\{ \begin{bmatrix} \tau_{23} \\ \tau_{34} \end{bmatrix} = \begin{bmatrix} F_{T3} \\ F_{T4} \end{bmatrix} + \begin{bmatrix} \frac{m_2+m_3}{m_2 m_3} & -\frac{1}{m_3} \\ -\frac{1}{m_3} & \frac{m_3+m_4}{m_3 m_4} \end{bmatrix}^{-1} \begin{bmatrix} \frac{-1}{m_2} \\ 0 \end{bmatrix} (F_{T4} - k_{12}\,\mathrm{sgn} z_2) \right.$$

If signals τ_{23} and τ_{34} satisfy the sliding mode conditions $|\tau_{23}| < k_{23}$ and $|\tau_{34}| < k_{34}$, the variables z_3 and z_4 are kept to zero and the dynamics of variable z_2 is described by the following equation:

$$\left\{ \dot{z}_2 = \begin{bmatrix} \frac{m_1+m_2}{m_1 m_2} & -\frac{1}{m_2} & 0 \end{bmatrix} \begin{bmatrix} F_{T2} - k_{12}\,\mathrm{sgn} z_2 \\ F_{T3} - \tau_{23} \\ F_{T4} - \tau_{34} \end{bmatrix} \right.$$

If the sliding mode conditions are not satisfied, a different configuration for the sliding variables must be considered (for instance $z_2 \neq 0$, $z_3 \neq 0$ and $z_4 = 0$). All the other cases must be treated in a similar way.

4 Simulation results

The parameters used in simulation are: $m_1 = m_2 = m_3 = m_4 = 10$ Kg; $k_{12} = 12$ N, $k_{23} = 10$ N, $k_{34} = 13$ N and ini-

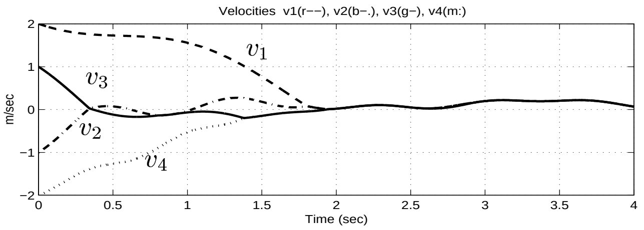

Figure 5: Velocities v_i of the masses, $i \in \{1, 2, 3, 4\}$.

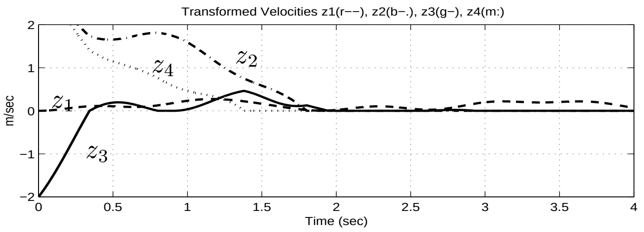

Figure 6: Transformed velocities z_i, $i \in \{1, 2, 3, 4\}$.

tial conditions $v(0) = [2, -1, 1, 2]$ m/s. The forces F_i applied to the masses are sinusoidal functions $F_i = A_i \sin(2\pi f_i t)$ with amplitudes $A_i = [11, 11, -8, 8]$ N and frequencies $f_i = [.45, 1.15, 0.85, 1.5]$ Hz. Note that the amplitudes A_i are lower than the friction coefficients $k_{i-1,i}$. This ensures that in finite time the four masses collapse to an unique mass. The velocities v_i obtained in simulations are shown in Fig. 5. Note that for $t \simeq 0.4$ s it happens that $v_2 = v_3$, but the exchanged torque τ_{23} is greater than the friction coefficient k_{23} and therefore masses m_2 and m_3 continue to slip. For $t \simeq 0.8$ s it happens that $v_2 = v_3$, but in this case $|\tau_{23}| < k_{23}$, so the two masses collapse together. This condition persists until about $t \simeq 1$ s, when the external forces and the inertia dynamic action overcome the coefficient k_{23}, so the two masses start slipping again. The transformed velocities z_i are shown in Fig. 6. The "Main Dynamics" is described by velocity z_1 while the "Relative Dynamics" are described by the other variables z_2, z_3 and z_4. Note that for $t \simeq 0.4$ s and 0.8 s $\leq t \leq$1 s, when $v_2 = v_3$, the corresponding transformed variable z_3 is equal to zero. The equivalent controls $\tau_{i-1,i}$ are shown in Fig. 7. They represent the torques that at each instant the masses reciprocally exchange. When the relative angular velocity z_i is equal to zero, the corresponding equivalent control $\tau_{i-1,i}$ is not saturated, and it belongs to range $[-k_{i-1,i}, k_{i-1,i}]$.

4.1 Approximated simulations: the sliding mode model

A simulation model for the considered system is the sliding model described by equation (3). A comparison between the simulation results obtained with this sliding model and the model proposed in section (2.2) is shown in Fig. 8. The matching is good except for obvious chattering phenomena on the output velocities v_i of the sliding approximated model. In the sliding model, the mean value of the switching action represents the torque that at each time is exchanged by two consecutive masses: $\tau_{i-1,i} = \overline{k_{i-1,i}\ \mathrm{sgn}(z_i)}$. This fact is clearly shown in the lower part of Fig. 9 where the equivalent control τ_{23} obtained with the proposed model is compared with the switching action filtered by a first order Butterworth filter.

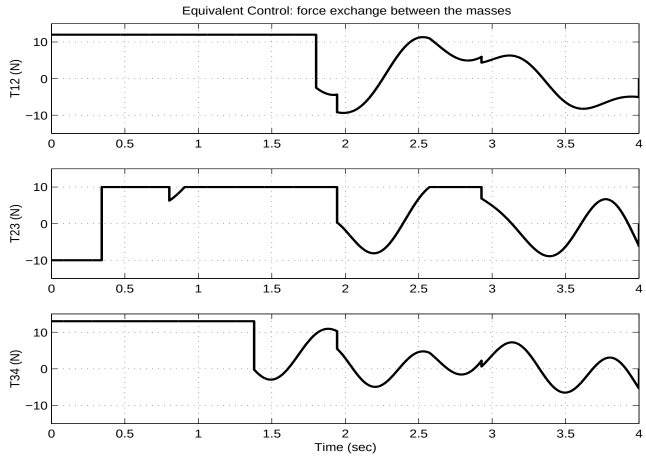

Figure 7: Equivalent controls $\tau_{i-1,i}(t)$, $i \in \{1, 2, 3, 4\}$.

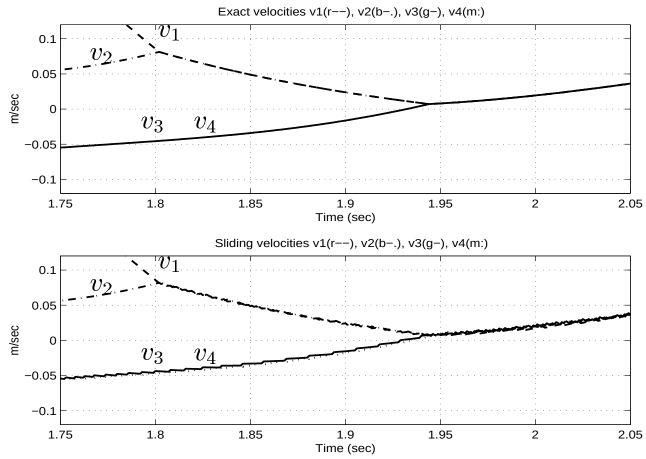

Figure 8: Comparison between the proposed simulation model and the sliding approximated model: the velocities v_i.

5 Clutch with Torsional Damper-Spring

Let us now consider the automotive transmission system shown in Fig. 10. The torsional damper-spring is an elastic element that often is inserted between the clutch disk and the primary of the gear shaft with the objective of filtering the torque spikes generated by the engine. Let J_1, J_2 and J_3 denote the inertias of the engine shaft, the torsional damper disk and the transmission shaft, respectively. The system simulation is critical due to the presence of coulomb frictions between the three inertias J_1, J_2 and J_3. The coulomb friction k_{23} is assumed to be constant while the coulomb friction k_{12} is supposed to be modulated by an external normal force F_n acting on the disk.

5.1 Dynamic Model of the clutch with the damper-spring

The differential equations describing the system are:

$$\begin{cases} J_1\dot{\omega}_1 = C_1 - b_1\,\omega_1 - k_{12}\,\mathrm{sgn}(\omega_1-\omega_2) \\ J_2\dot{\omega}_2 = -b_2\omega_2 + k_{12}\mathrm{sgn}(\omega_1-\omega_2) - k_{23}\mathrm{sgn}(\omega_2-\omega_3) - \phi_e(\theta_d) \\ J_3\dot{\omega}_3 = C_3 - b_3\,\omega_3 + k_{23}\,\mathrm{sgn}(\omega_2-\omega_3) + \phi_e(\theta_d) \\ \dot{\theta}_d = \omega_2 - \omega_3 \end{cases}$$

where C_1, C_3 are the engine torque and the resistant external torque; ω_1, ω_2 and ω_3 are the engine angular velocity, the tor-

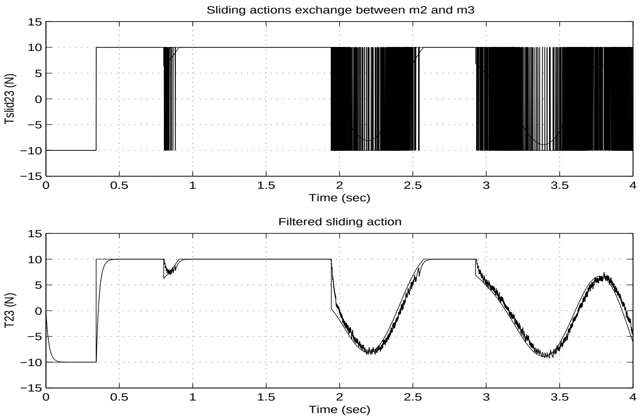

Figure 9: Equivalent control τ_{23} compared with the mean value of the sliding actions.

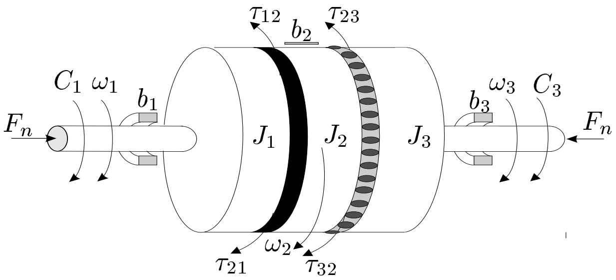

Figure 10: The clutch with the torsional damper-spring.

sional damper disk angular velocity and the transmission shaft angular velocity, respectively; b_1, b_2 and b_3 are the viscous friction coefficients; k_{12} and k_{23} are the coulomb friction coefficients; $\phi = \phi_e(\theta_d)$ is the elastic torque of the torsional damper-spring which is a function of the relative position $\theta_d = \theta_2 - \theta_3$ between the two inertias J_2 and J_3.

5.2 Simulation results

The parameters used in simulation are: $J_1 = 9$ Kg m^2, $J_2 = 5$ Kg m^2, $J_3 = 13$ Kg m^2, $b_1 = 12$ N m s/rad, $b_2 = 1$ N m s/rad, $b_3 = 7$ N m s/rad, $k_{23} = 50$ N m and $C_3 = 0$. The time behaviours of the coulomb friction k_{12} and the torque C_1 are shown in Fig. 11. The angular velocities $\omega_1(t)$, $\omega_2(t)$ and $\omega_3(t)$

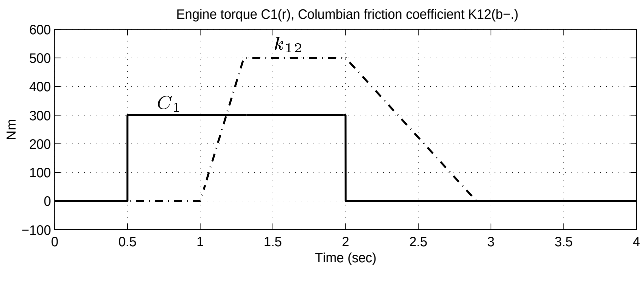

Figure 11: Time behaviours of the coulomb friction k_{12} and torque C_1.

are shown in Fig. 12. At time $t = 0.5$ s the velocity ω_1 starts to increase because the torque $C_1 = 300$ N m is applied; the velocities ω_1 and ω_2 remain at zero because the coefficient k_{12} is zero. At time $t = 1$ s, when the coefficient k_{12} starts to increase, the velocity ω_1 decreases and velocities ω_2 and ω_3 start

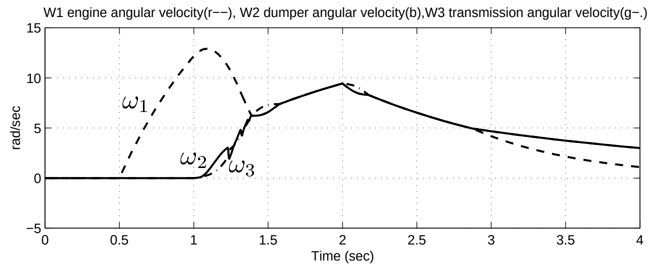

Figure 12: Angular velocities $\omega_1(t)$, $\omega_2(t)$ and $\omega_3(t)$.

to increase. At time $t \simeq 1.6$ s all the inertias move together at the same angular velocity $\omega_1(t) = \omega_2(t) = \omega_3(t)$. Note that $\omega_3(t)$ is smoother that $\omega_2(t)$ due to the presence of the torsional damper-spring. When the coefficient k_{12} decreases to zero the angular velocity ω_1 starts to differ again from ω_2 and ω_3.

6 Conclusions

In this paper, the problem of finding a simple and effective simulation model for a class of Variable Dynamic Dimension Systems (VDDS) composed by n masses that reciprocally slip and interact by means of the coulomb friction, has been presented. This type of systems is very common in mechanics. For simulating this type of systems, in literature one can found huge and complex models, or approximated models that give rough results. In this paper, a particular space state transformation that puts in evidence the main dynamics and the $n-1$ relative dynamics of the considered system, has been presented. Due to this new point of view, the exact simulation of the considered VDDS is easier and faster. Finally, the proposed simulation model has been applied to a VDDS system of interest in automotive applications: the clutch with "torsional damper".

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