System and Control Theory
Test of February 9, 2017
Questions and Exercises

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1. For a time-variant continuous-time linear system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{x}(t)$, the state transition matrix $\mathbf{\Phi}(t, t_0)$ is solution of the following matrix differential equation:

$$\frac{d}{dt}\mathbf{\Phi}(t, t_0) = \dots \qquad , \qquad \mathbf{\Phi}(t_0, t_0) = \dots$$

2. Write the general solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ starting from the initial condition $\mathbf{x}(0)$ at time $h = 0$:

$$\mathbf{x}(k) = \dots$$

3. Write the time behavior of the output function $\mathbf{y}(t)$, solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and the static equation $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$ at time t_0:

$$\mathbf{y}(t) =$$

4. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \quad \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \quad \end{bmatrix}$$

The system is: ○ reachable ○ not-reachable ○ observable ○ not-observable

5. The following symbolic representation:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), 0, k) \\ \mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), 0, k) \end{cases} \qquad \mathbf{x}(k) \in \mathbf{R}^n$$

is used for describing a system with the following characteristics:

- ○ a linear system;
- ○ a lumped system;
- ○ a dynamic system;
- ○ a system without inputs;
- ○ a time-varying system;
- ○ a continuous-time system;

6. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^{\mathrm{T}}$, the free evolution of the following continuous-time autonomous system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) \qquad \mathbf{x}(t) = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

7. Provide the continuous and discrete-time solutions of the following two mathematical expressions involving the Laplace transform and the $\mathcal{Z}$-transform, respectively:

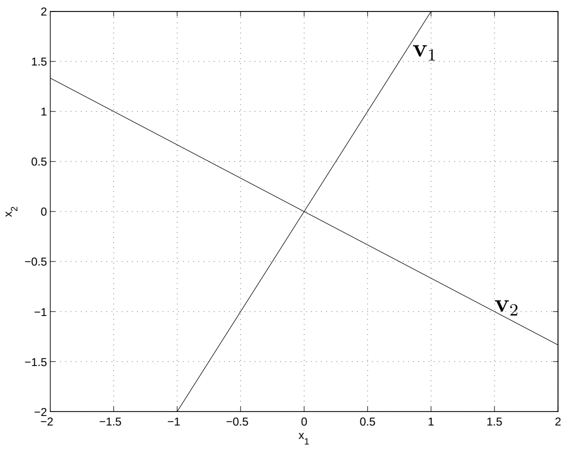
$$\mathcal{L}^{-1}[(s\mathbf{I}-\mathbf{A})^{-1}] = \qquad\qquad \mathcal{Z}^{-1}[z(z\mathbf{I}-\mathbf{A})^{-1}] =$$

8. Given a discrete-time linear system: $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ and $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$. Apply the $\mathcal{Z}$-transform to the given system and provide the expression of the transform $\mathbf{y}(z)$ of the output vector $\mathbf{y}(k)$ corresponding to the *forced evolution* of the system:

$$\mathbf{y}(z) =$$

9. Draw qualitatively the trajectories of a second order dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ characterized by the eigenvalues λ_i and the eigenvectors $\mathbf{v}_i$ shown in the two following boxes.

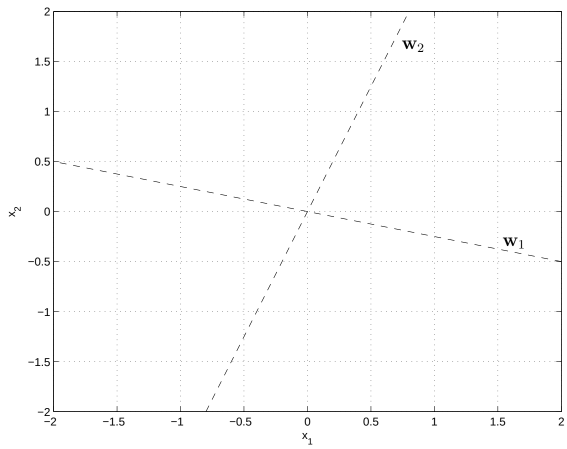
1) Eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -3$. The corresponding real eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are shown in the figure.



◯ Node? ◯ Focus? ◯ Saddle?

◯ Stable? ◯ Unstable?

2) Eigenvalues: $\lambda_{1,2} = 1 \pm 2j$. Eigenvectors: $\mathbf{v}_1 = \mathbf{w}_1 + \mathbf{w}_2 j$ and $\mathbf{v}_2 = \mathbf{v}_1^*$. The real vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are shown in the figure.



◯ Node? ◯ Focus? ◯ Saddle?

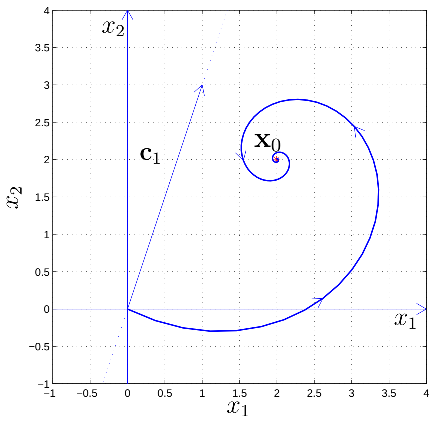
◯ Stable? ◯ Unstable?

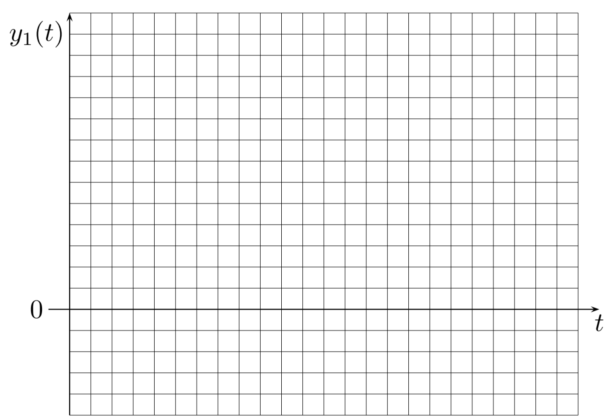
10. Consider the following linear dynamic system: $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$ and $\mathbf{y} = \mathbf{c}_1\,\mathbf{x}$.

a) Write the formula used for computing the equilibrium point $\mathbf{x}_0$ corresponding to the constant input $\mathbf{u} = \mathbf{u}_0$:

$$\mathbf{x}_0 = \qquad\qquad\qquad .$$

b) Given the system trajectory shown below in the left box, draw the qualitative behavior of the time response of the system output $y(t)$:





11. Given the dynamic system shown below, write the transfer function $G(s)$ which links the input $u(t)$ to the output $y(t)$:

$$\left\{\begin{array}{rcl} \dot{\mathbf{x}}(t) & = & \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 5 \\ 1 \\ 2 \\ 3 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) \end{array}\right. \qquad G(s) =$$

Without making specific calculations it can be stated that surely:

- ◯ the system is reachable;
- ◯ the system is stable;
- ◯ the system is observable;
- ◯ the system has zero relative degree;

12. Write the structure of the matrix $\mathbf{P}^{-1}$ of the state space transformation $\mathbf{x} = \mathbf{P}\overline{\mathbf{x}}$ which brings a not-observable system in the standard observability form:

$$\mathbf{P}^{-1} =$$

Moreover, write the block structure of the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$:

$$\overline{\mathbf{A}} = \begin{bmatrix} \quad \end{bmatrix}, \qquad \overline{\mathbf{B}} = \begin{bmatrix} \quad \end{bmatrix}, \qquad \overline{\mathbf{C}} = \begin{bmatrix} \quad \end{bmatrix}$$

Write the simplified form of transfer matrix $\mathbf{H}(s)$ of the system $\mathcal{S}$ as a function of the sub-matrices $\mathbf{A}_{i,j}$, $\mathbf{B}_i$ and $\mathbf{C}_j$ that characterize the system $\bar{\mathcal{S}} = (\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\mathbf{C}})$:

$$\mathbf{H}(s) =$$

13. Given a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{u}$ completely reachable and with only one input. Let $\Delta_{\mathbf{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$ be the characteristic polynomial of matrix $\mathbf{A}$ and let $p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0$ be a monic polynomial freely chosen. Write the expression of vector $\mathbf{k}^{\mathrm{T}}$ which using the static feedback $\mathbf{u} = \mathbf{k}^{\mathrm{T}}\mathbf{x}$ is able to match the eigenvalues of matrix $\mathbf{A} + \mathbf{b}\mathbf{k}^{\mathrm{T}}$ with the roots of polynomial $p(\lambda)$:

$$\mathbf{k}^{\mathrm{T}} =$$

where $\mathbf{k}_c^{\mathrm{T}} = [\qquad\qquad]$.

14. A system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is "stabilizable" using a static state feedback

- ◯ if the system is stable;
- ◯ if the unstable part of the system is reachable;
- ◯ if the system is observable;
- ◯ if the part not-reachable of the system is stable;

15. Write the "*separation property*" of the regulator:

16. Write the explicit form of the Ackermann formula for computing the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of a feedback system:

$$\mathbf{k}^{\mathrm{T}} =$$

17. For the discrete time-invariant linear system $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,u(k)$, the input sequence $\mathbf{u}(0), \ldots, \mathbf{u}(\bar{k}-1)$ which moves the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(\bar{k})$:

- ○ exists if the system is reachable;
- ○ exists if $\mathbf{x}(\bar{k}) \in \mathcal{X}^{+}(\bar{k})$;
- ○ exists if $\mathbf{x}(\bar{k}) - e^{\mathbf{A}\bar{k}}\mathbf{x}(0) \in \mathcal{X}^{+}(\bar{k})$;
- ○ exists if $\mathbf{x}(\bar{k}) - \mathbf{A}^{\bar{k}}\mathbf{x}(0) \in \mathcal{X}^{+}(\bar{k})$;

18. An "open loop" state estimator can be used:

- ○ if the system is reachable;
- ○ if the system is observable;
- ○ if the system is asymptotically stable;
- ○ if the unstable part of the system is observable;

19. Given the discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, write the structure of a "full order closed loop" state estimator and the time evolution of the corresponding estimation error $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ starting from the initial condition $\mathbf{e}(0)$:

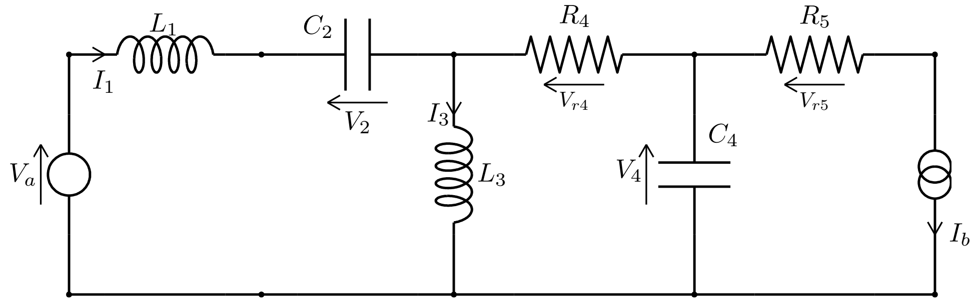
$\hat{\mathbf{x}}(k+1) =$ $\qquad\qquad\qquad$ $\mathbf{e}(k) =$

20. Write the La Salle - Krasowskii stability criterion for discrete-time nonlinear systems.

Consider the discrete-time nonlinear system $\mathbf{x}(k{+}1) = \mathbf{f}(\mathbf{x}(k),\, \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.

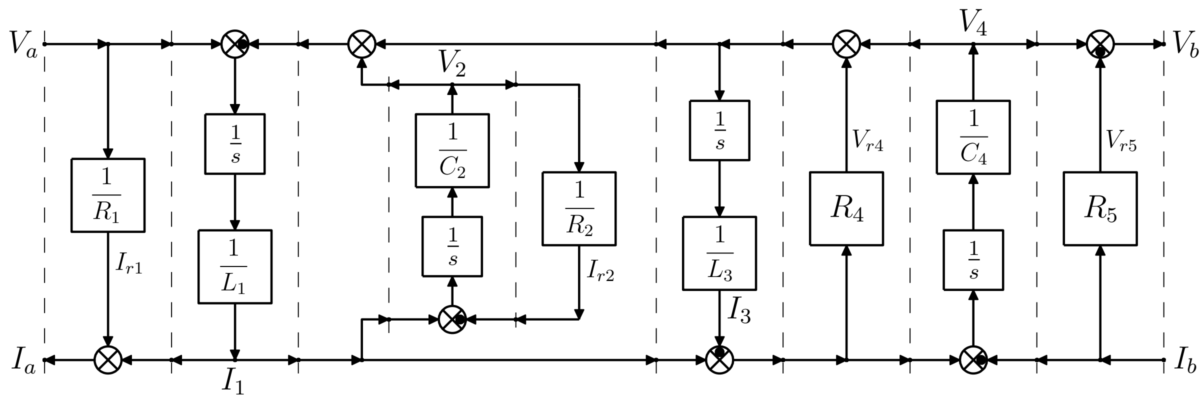
If ...

21. Consider the following electric circuit composed by inductances L_1, L_3, capacities C_2, C_4 and resistances R_4 and R_5. The system has two inputs: the voltage V_a and the current I_b. The outputs of the system are: the current I_a and the voltage V_b.



Write the POG model of the given electric circuit:

22. Consider the following POG scheme which describes the dynamics of an electric circuit composed by inductances L_1, L_3, capacities C_2, C_4 and resistances R_1, R_2, R_4 and R_5:



Let $\mathbf{x} = \begin{bmatrix} I_1 & V_2 & I_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

23. A POG dynamic system $\mathbf{S}$ can be transformed using a ***"congruent"*** transformation $\mathbf{x} = \mathbf{T}\,\mathbf{z}$:

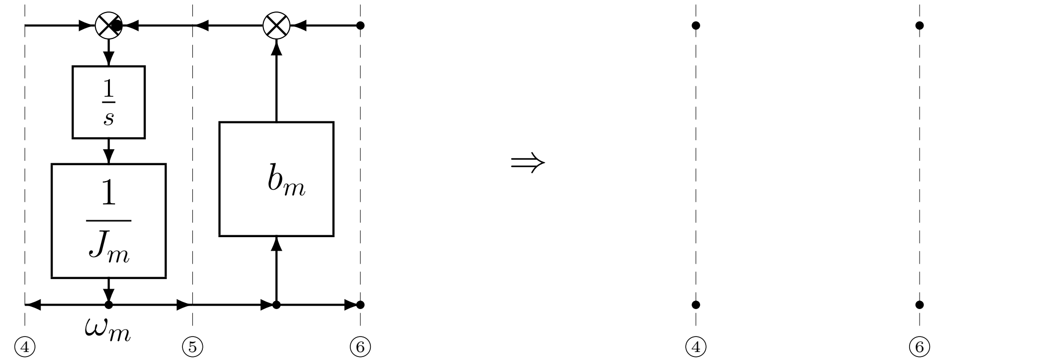
$$\mathbf{S} = \begin{cases} \mathbf{L}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u} \end{cases} \quad \overset{\mathbf{x}=\mathbf{T}\,\mathbf{z}}{\Longrightarrow} \quad \overline{\mathbf{S}} = \begin{cases} \overline{\mathbf{L}}\,\dot{\mathbf{z}} = \overline{\mathbf{A}}\,\mathbf{z} + \overline{\mathbf{B}}\,\mathbf{u} \\ \mathbf{y} = \overline{\mathbf{C}}\,\mathbf{z} + \mathbf{D}\,\mathbf{u} \end{cases}$$

where $\overline{\mathbf{S}}$ is the transformed system. Write the expressions of the transformed "**congruent**" matrices $\overline{\mathbf{L}}$, $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$ as a function of the given system matrices $\mathbf{L}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$:

$\overline{\mathbf{L}} =$ $\qquad\qquad$ $\overline{\mathbf{A}} =$ $\qquad\qquad$ $\overline{\mathbf{B}} =$ $\qquad\qquad$ $\overline{\mathbf{C}} =$

24. When $J_m = 0$ the POG scheme shown in the lower left can be graphically reduced. Write in the lower right the equivalent transformed and reduced POG scheme when $J_m = 0$:

POG scheme to be reduced $\qquad\qquad$ Reduced POG scheme when $J_m = 0$



25. Given the following autonomous discrete-time nonlinear system:

$$\begin{cases} x_1(k+1) &= x_1(k)\cos x_2(k) + 2\,x_2^3(k) \\ x_2(k+1) &= \cos x_2(k) - 1 + \alpha x_1(k)\sin x_2(k) \end{cases}$$

a) Verify if the point $\mathbf{x}_0 = (1, 0)$ is an equilibrium point for the given system:

b) Linearize the system in the neighborhood of point $\mathbf{x}_0 = (1, 0)$ computing the matrix $\mathbf{A}$ of the corresponding linearized system:

c) Study, for varying α, the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = (1, 0)$ using the reduced Lyapunov criterion:

26. Given the following continuous-time autonomous nonlinear system:

$$\begin{cases} \dot{x}_1 &= -2x_1^3 - 2x_2^4 \\ \dot{x}_2 &= x_1 x_2 - x_2^3 \end{cases}$$

Study the stability of the nonlinear system in the vicinity of the origin $(x_1,\, x_2) = (0,\, 0)$ using the "direct" Lyapunov criterion and the following positive definite function: $V(\mathbf{x}) = x_1^2 + x_2^4$.

27. Given the nonlinear discrete system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$ and the following function $V(\mathbf{x}(k))$:

$$\begin{cases} x_1(k+1) &= x_1 x_2 \\ x_2(k+1) &= x_1 - x_2 \end{cases} \qquad V(\mathbf{x}(k)) = x_1^2 + x_2^2$$

a) Compute the two equilibrium points $\mathbf{x}_{e1} and\ \mathbf{x}_{e2}$ of the given discrete system:

$\mathbf{x}_{e1} = (\quad,\quad)$ $\qquad\qquad$ $\mathbf{x}_{e2} = (\quad,\quad)$

b) Compute the function $\Delta V(\mathbf{x}(k))$ used in the direct Lyapunov criterion:

$\Delta V(\mathbf{x}(k)) =$