

System and Control Theory
Test of February 9, 2017
Questions and Exercises

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1. For a time-variant continuous-time linear system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$, the state transition matrix $\Phi(t, t_0)$ is solution of the following matrix differential equation:

$$\frac{d}{dt}\Phi(t, t_0) = \mathbf{A}(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = \mathbf{I}$$

2. Write the general solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ starting from the initial condition $\mathbf{x}(0)$ at time $h = 0$:

$$\mathbf{x}(k) = \mathbf{A}^k \mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{(k-j-1)} \mathbf{B} \mathbf{u}(j)$$

3. Write the time behavior of the output function $\mathbf{y}(t)$, solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and the static equation $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$ at time t_0 :

$$\mathbf{y}(t) = \mathbf{C} e^{\mathbf{A}(t-t_0)} \mathbf{x}(t_0) + \mathbf{C} \int_{t_0}^t e^{\mathbf{A}(t-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau + \mathbf{D} \mathbf{u}(t)$$

4. Compute the reachability matrix \mathcal{R}^+ and the observability matrix \mathcal{O}^- of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \quad \mathcal{R}^+ = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The system is: reachable not-reachable observable not-observable

5. The following symbolic representation:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), 0, k) \\ \mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), 0, k) \end{cases} \quad \mathbf{x}(k) \in \mathbf{R}^n$$

is used for describing a system with the following characteristics:

- a linear system; a lumped system;
 a dynamic system; a system with zero inputs;
 a time-varying system; a continuous-time system;

6. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0), x_2(0), x_3(0), x_4(0)]^T$, the free evolution of the following continuous-time autonomous system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) \quad \mathbf{x}(k) = \begin{bmatrix} e^t & t e^t & \frac{t^2}{2} e^t & \frac{t^3}{6} e^t \\ 0 & e^t & t e^t & \frac{t^2}{2} e^t \\ 0 & 0 & e^2 & t e^t \\ 0 & 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

7. Provide the continuous and discrete-time solutions of the following two mathematical expressions involving the Laplace transform and the \mathcal{Z} -transform, respectively:

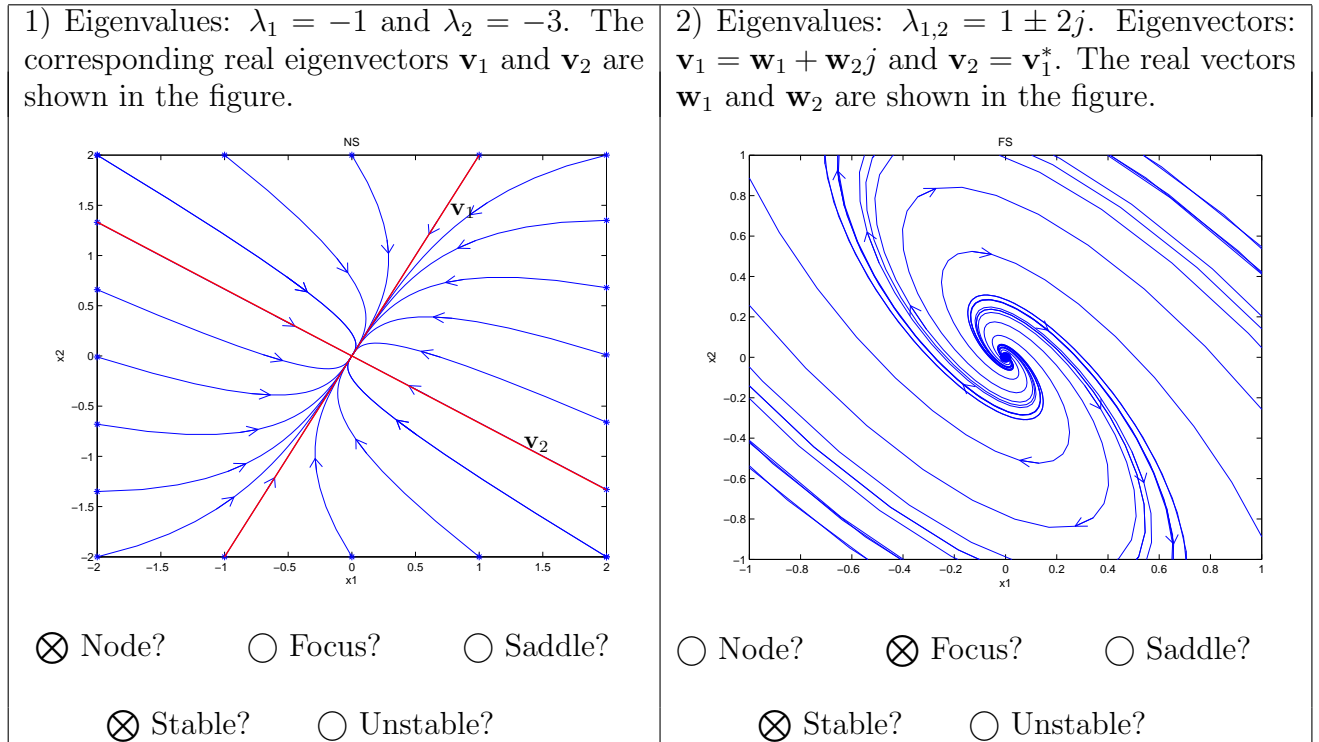
$$\mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = e^{\mathbf{A}t}$$

$$\mathcal{Z}^{-1}[z(z\mathbf{I} - \mathbf{A})^{-1}] = \mathbf{A}^k$$

8. Given a discrete-time linear system: $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ and $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$. Apply the \mathcal{Z} -transform to the given system and provide the expression of the transform $\mathbf{y}(z)$ of the output vector $\mathbf{y}(k)$ corresponding to the *forced evolution* of the system:

$$\mathbf{y}(z) = [\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{u}(z)$$

9. Draw qualitatively the trajectories of a second order dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ characterized by the eigenvalues λ_i and the eigenvectors \mathbf{v}_i shown in the two following boxes.

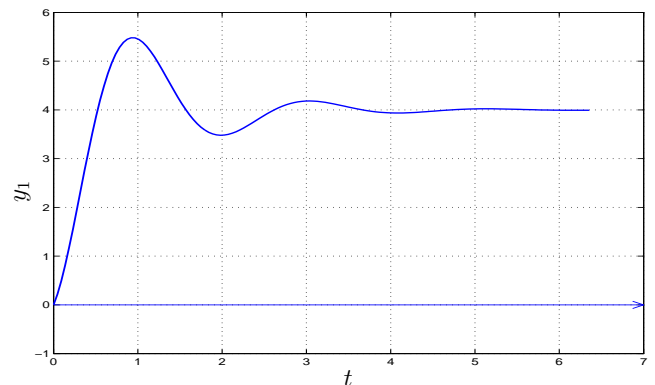
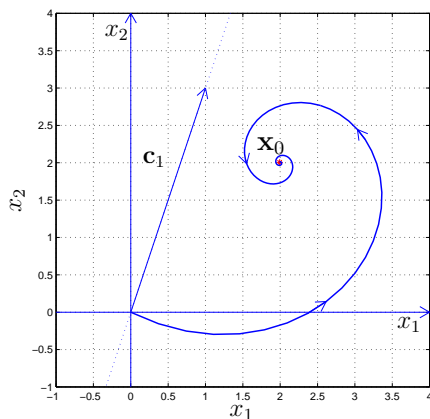


10. Consider the following linear dynamic system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{c}_1\mathbf{x}$.

- a) Write the formula used for computing the equilibrium point \mathbf{x}_0 corresponding to the constant input $\mathbf{u} = \mathbf{u}_0$:

$$\mathbf{x}_0 = -\mathbf{A}^{-1}\mathbf{B}\mathbf{u}_0.$$

- b) Given the system trajectory shown below in the left box, draw the qualitative behavior of the time response of the system output $y(t)$:



11. Given the dynamic system shown below, write the transfer function $G(s)$ which links the input $u(t)$ to the output $y(t)$:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 5 \\ 1 \\ 2 \\ 3 \end{bmatrix} u(t) \\ y(t) = [0 \ 0 \ 0 \ 1] \mathbf{x}(t) \end{cases} \quad G(s) = \frac{3s^3 + 2s^2 + s + 5}{s^4 + s^3 + 6s^2 + 3s + 4}$$

Without making specific calculations it can be stated that surely:

- the system is reachable; the system is stable;
 the system is observable; the system has zero relative degree;
12. Write the structure of the matrix \mathbf{P}^{-1} of the state space transformation $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ which brings a not-observable system in the standard observability form:

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} \quad \text{where} \quad \text{Im}\mathbf{P}_1^T = \text{Im}(\mathcal{O}^-)^T \text{ and } \mathbf{P}_2 \text{ makes non singular the matrix } \mathbf{P}^{-1}.$$

Moreover, write the block structure of the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$:

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{0} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix} \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

$$\bar{\mathbf{C}} = [\mathbf{C}_1 \ \mathbf{0}]$$

Write the simplified form of transfer matrix $\mathbf{H}(s)$ of the system \mathcal{S} as a function of the submatrices $\mathbf{A}_{i,j}$, \mathbf{B}_i and \mathbf{C}_j that characterize the system $\bar{\mathcal{S}} = (\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$:

$$\mathbf{H}(s) = \mathbf{C}_1(s\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{B}_1$$

13. Given a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$ completely reachable and with only one input. Let $\Delta_{\mathbf{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_1\lambda + \alpha_0$ be the characteristic polynomial of matrix \mathbf{A} and let $p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \dots + d_1\lambda + d_0$ be a monic polynomial freely chosen. Write the expression of vector \mathbf{k}^T which using the static feedback $u = \mathbf{k}^T\mathbf{x}$ is able to match the eigenvalues of matrix $\mathbf{A} + \mathbf{b}\mathbf{k}^T$ with the roots of polynomial $p(\lambda)$:

$$\mathbf{k}^T = \mathbf{k}_c^T \left\{ \left[\mathbf{b}, \mathbf{A}\mathbf{b}, \dots, \mathbf{A}^{n-1}\mathbf{b} \right] \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} & 1 \\ \alpha_2 & \dots & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \right\}^{-1}$$

where $\mathbf{k}_c^T = [\alpha_0 - d_0, \alpha_1 - d_1, \dots, \alpha_{n-1} - d_{n-1}]$.

14. A system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is “stabilizable” using a static state feedback

- if the system is stable; if the unstable part of the system is reachable;
 if the system is observable; if the part not-reachable is asymptotically stable;

15. Write the “*separation property*” of the regulator:

The design of the feedback block $(\mathbf{A} + \mathbf{B}\mathbf{K})$ and the estimation block $(\mathbf{A} + \mathbf{L}\mathbf{C})$ can be done independently:

$$\det[z\mathbf{I} - \bar{\mathbf{A}}] = \det[z\mathbf{I} - (\mathbf{A} + \mathbf{B}\mathbf{K})] \det[z\mathbf{I} - (\mathbf{A} + \mathbf{L}\mathbf{C})]$$

16. Write the explicit form of the Ackermann formula for computing the vector \mathbf{k}^T allowing the free positioning of the eigenvalues of a feedback system:

$$\mathbf{k}^T = - [0 \ \dots \ 0 \ 1] (\mathcal{R}^+)^{-1} p(\mathbf{A})$$

17. For the discrete time-invariant linear system $\mathbf{x}(k+1) = \mathbf{A} \mathbf{x}(k) + \mathbf{B} u(k)$, the input sequence $\mathbf{u}(0), \dots, \mathbf{u}(\bar{k}-1)$ which moves the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(\bar{k})$:

- exists if the system is reachable; exists if $\mathbf{x}(\bar{k}) - e^{\mathbf{A}\bar{k}}\mathbf{x}(0) \in \mathcal{X}^+(\bar{k})$;
 exists if $\mathbf{x}(\bar{k}) \in \mathcal{X}^+(\bar{k})$; exists if $\mathbf{x}(\bar{k}) - \mathbf{A}^{\bar{k}}\mathbf{x}(0) \in \mathcal{X}^+(\bar{k})$;

18. An “open loop” state estimator can be used:

- if the system is reachable; if the system is asymptotically stable;
 if the system is observable; if the unstable part of the system is observable;

19. Given the discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, write the structure of a “full order closed loop” state estimator and the time evolution of the corresponding estimation error $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ starting from the initial condition $\mathbf{e}(0)$:

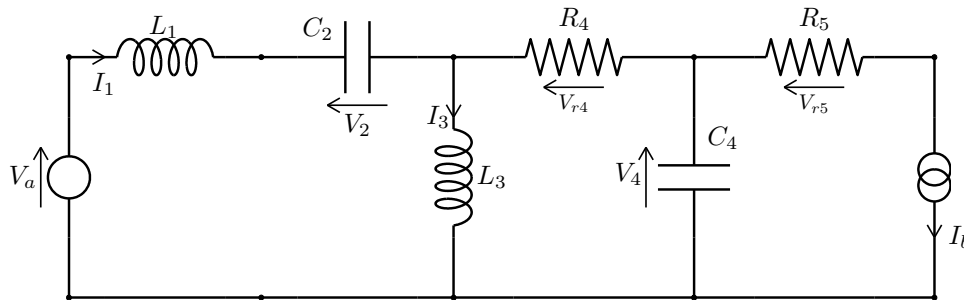
$$\hat{\mathbf{x}}(k+1) = (\mathbf{A} + \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{L}\mathbf{y}(k), \quad \mathbf{e}(k) = (\mathbf{A} + \mathbf{L}\mathbf{C})^k \mathbf{e}(0)$$

20. Write the La Salle - Krasowskii stability criterion for discrete-time nonlinear systems.

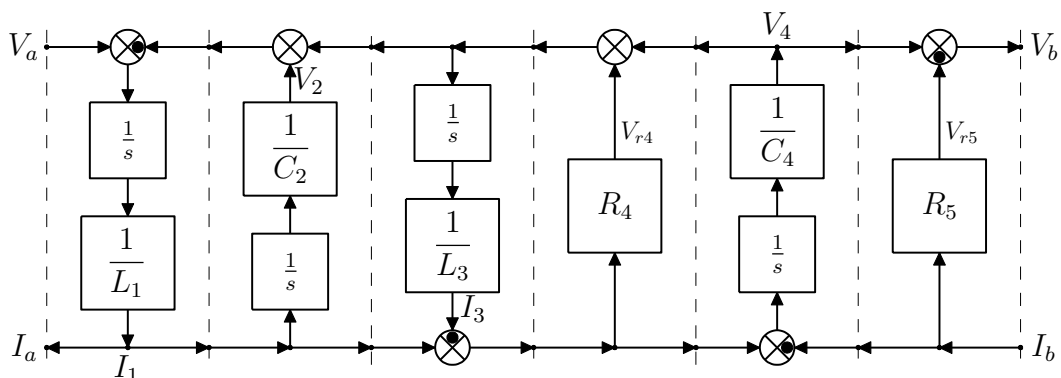
Consider the discrete-time nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}_0)$ and let \mathbf{x}_0 be an equilibrium point corresponding to the constant input \mathbf{u}_0 .

If in a neighborhood W of point \mathbf{x}_0 it exists a continuous function $V(\mathbf{x}) : W \rightarrow \mathcal{R}$ positive definite, if the function $\Delta V(\mathbf{x})$ is negative semidefinite and if the set $\mathcal{N} = \{\mathbf{x} \in W | \Delta V(\mathbf{x}) = 0\}$ does not contain perturbed trajectories of the given system, then \mathbf{x}_0 is an asymptotically stable equilibrium point.

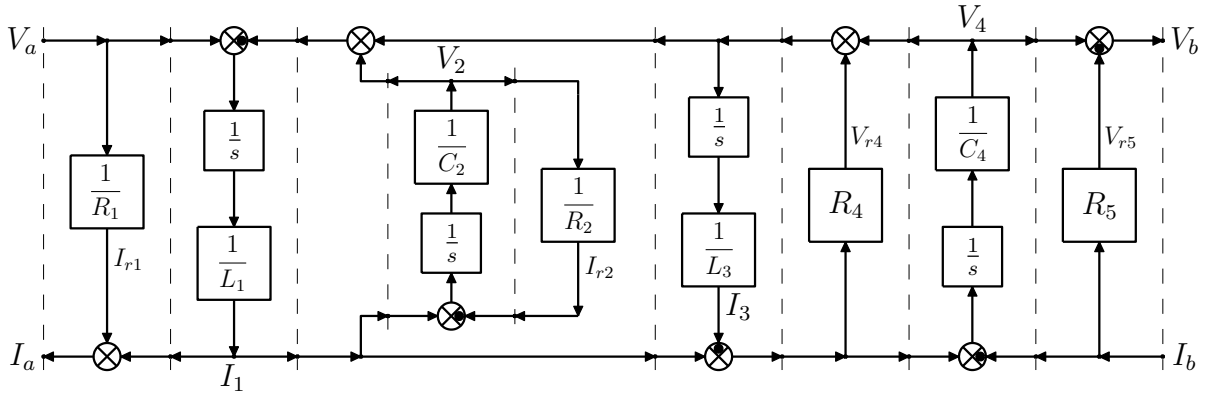
21. Consider the following electric circuit composed by inductances L_1, L_3 , capacities C_2, C_4 and resistances R_4 and R_5 . The system has two inputs: the voltage V_a and the current I_b . The outputs of the system are: the current I_a and the voltage V_b .



Write the POG model of the given electric circuit:



22. Consider the following POG scheme which describes the dynamics of an electric circuit composed by inductances L_1, L_3 , capacities C_2, C_4 and resistances R_1, R_2, R_4 and R_5 :



Let $\mathbf{x} = [I_1 \ V_2 \ I_3 \ V_4]^T$ be the state vector, $\mathbf{u} = [V_a \ I_b]^T$ the input vector and $\mathbf{y} = [I_a \ V_b]^T$ the output vector. Write the corresponding dynamic system $\bar{\mathbf{L}}\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \bar{\mathbf{C}}\mathbf{x} + \bar{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\bar{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_4 & -1 & R_4 & -1 \\ 1 & -\frac{1}{R_2} & 0 & 0 \\ R_4 & 0 & -R_4 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}}_{\bar{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\bar{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\bar{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & -R_5 \end{bmatrix}}_{\bar{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

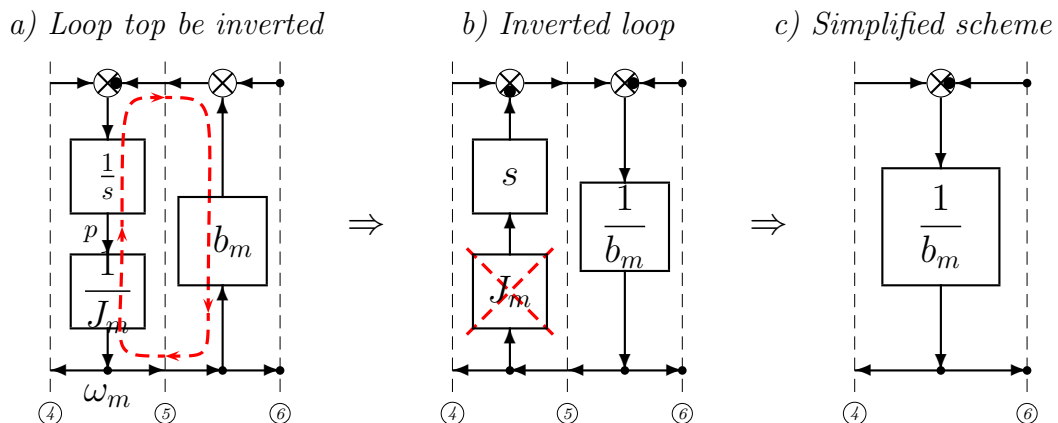
23. A POG dynamic system \mathbf{S} can be transformed using a “congruent” transformation $\mathbf{x} = \mathbf{T}\mathbf{z}$:

$$\mathbf{S} = \begin{cases} \mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \xrightarrow{\mathbf{x}=\mathbf{T}\mathbf{z}} \bar{\mathbf{S}} = \begin{cases} \bar{\mathbf{L}}\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}\mathbf{u} \\ \mathbf{y} = \bar{\mathbf{C}}\mathbf{z} + \mathbf{D}\mathbf{u} \end{cases}$$

where $\bar{\mathbf{S}}$ is the transformed system. Write the expressions of the transformed “congruent” matrices $\bar{\mathbf{L}}$, $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$ as a function of the given system matrices \mathbf{L} , \mathbf{A} , \mathbf{B} and \mathbf{C} :

$$\bar{\mathbf{L}} = \mathbf{T}^T \mathbf{L} \mathbf{T}, \quad \bar{\mathbf{A}} = \mathbf{T}^T \mathbf{A} \mathbf{T}, \quad \bar{\mathbf{B}} = \mathbf{T}^T \mathbf{B}, \quad \bar{\mathbf{C}} = \mathbf{C} \mathbf{T}$$

24. When $J_m = 0$ the POG scheme shown in the lower left can be graphically reduced. Write in the lower right the equivalent transformed and reduced POG scheme when $J_m = 0$:



25. Given the following autonomous discrete-time nonlinear system:

$$\begin{cases} x_1(k+1) = x_1(k) \cos x_2(k) + 2x_2^3(k) \\ x_2(k+1) = \cos x_2(k) - 1 + \alpha x_1(k) \sin x_2(k) \end{cases}$$

a) Verify if the point $\mathbf{x}_0 = (1, 0)$ is an equilibrium point for the given system:

$\mathbf{x}_0 = (1, 0)$ is an equilibrium point for the system because the following relations hold:

$$\begin{cases} x_1(k) = x_1(k) \cos x_2(k) + 2x_2^3(k) \\ x_2(k) = \cos x_2(k) - 1 + \alpha x_1(k) \sin x_2(k) \end{cases} \rightarrow \begin{cases} 1 = 0 + 1 \\ 0 = 1 - 1 + 0 \end{cases}$$

b) Linearize the system in the neighborhood of point $\mathbf{x}_0 = (1, 0)$ computing the matrix \mathbf{A} of the corresponding linearized system:

The matrix \mathbf{A} of the linearized system has the following structure:

$$\mathbf{A} = \begin{bmatrix} \cos x_2 & -x_1 \sin x_2 + 6x_2^2 \\ \alpha \sin x_2 & -\sin x_2 + \alpha x_1 \cos x_2 \end{bmatrix}_{(x_1=1, x_2=0)} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$$

c) Study, for varying α , the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = (1, 0)$ using the reduced Lyapunov criterion:

The eigenvalues of matrix \mathbf{A} are the roots of following characteristic polynomial:

$$\det[z\mathbf{I}_2 - \mathbf{A}] = z^2 - (\alpha + 1)z + \alpha = (z - 1)(z - \alpha) = 0$$

The roots of the characteristic polynomial are: $z_1 = 1$ and $z_2 = \alpha$. Using the reduced Lyapunov criterion it is possible to state that: 1) for $|\alpha| > 1$ the nonlinear system is unstable in the neighborhood of point $\mathbf{x}_0 = (1, 0)$ because the eigenvalue $z_2 = \alpha$ is located outside the unit cycle; 2) for $|\alpha| \leq 1$ the reduced Lyapunov criterion cannot be used because the eigenvalue $z_1 = 1$ is located on the unit cycle

26. Given the following continuous-time autonomous nonlinear system:

$$\begin{cases} \dot{x}_1 = -2x_1^3 - 2x_2^4 \\ \dot{x}_2 = x_1x_2 - x_2^3 \end{cases}$$

Study the stability of the nonlinear system in the vicinity of the origin $(x_1, x_2) = (0, 0)$ using the “direct” Lyapunov criterion and the following positive definite function: $V(\mathbf{x}) = x_1^2 + x_2^4$.

Computing the time derivative of function $V(\mathbf{x})$ along the system’s trajectories one obtains:

$$\dot{V} = 2x_1(-2x_1^3 - 2x_2^4) + 4x_2^3(x_1x_2 - x_2^3) = -4x_1^4 - 4x_2^6 < 0$$

Applying the “direct” Lyapunov criterion it can be stated that in the neighborhood of the origin $\mathbf{x}_0 = 0$ the nonlinear system is asymptotically stable.

27. Given the nonlinear discrete system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$ and the following function $V(\mathbf{x}(k))$:

$$\begin{cases} x_1(k+1) = x_1x_2 \\ x_2(k+1) = x_1 - x_2 \end{cases} \quad V(\mathbf{x}(k)) = x_1^2 + x_2^2$$

a) Compute the two equilibrium points \mathbf{x}_{e1} and \mathbf{x}_{e2} of the given discrete system:

$$\mathbf{x}_{e1} = (0, 0) \quad \mathbf{x}_{e2} = (2, 1)$$

b) Compute the function $\Delta V(\mathbf{x}(k))$ used in the direct Lyapunov criterion:

$$\Delta V(\mathbf{x}(k)) = (x_1x_2)^2 + (x_1 - x_2)^2 - x_1^2 - x_2^2 = x_1^2x_2^2 - 2x_1x_2.$$