System and Control Theory
Test of Jenuary 23, 2017
Questions and Exercises

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1. Write the explicit form of the *transition matrix* $\mathbf{\Phi}(k,h)$ of the linear time-variant system $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$:

$$\mathbf{\Phi}(k,h) = \begin{cases} \mathbf{A}(k-1)\dots\mathbf{A}(h+1)\mathbf{A}(h) & \text{if } k > h \\ \mathbf{I} \text{ (Identity matrix)} & \text{if } k = h \end{cases}$$

2. Write the explicit solution $\mathbf{x}(k)$ of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ being $\mathbf{x}(h)$ the initial state at time h.

$$\mathbf{x}(k) = \mathbf{A}^{k-h}\mathbf{x}(h) + \sum_{j=h}^{k-1} \mathbf{A}^{k-j-1}\mathbf{B}\,\mathbf{u}(j)$$

3. Write the general solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(0)$ at time $t_0 = 0$:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

4. Give the meaning, for discrete linear systems, of the symbol $\mathcal{E}^-(k)$:

   $\mathcal{E}^-(k)$ is the set of the states which are non observable in k steps*, that is the states that are compatible with zero input and output sequences* $\mathbf{u}(\tau)$ *and* $\mathbf{y}(\tau)$ *for* $\tau = [0,\ 1, \dots, k-1]$.

5. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \mathcal{O}^- = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

   The system is: ⊗ reachable ◯ not-reachable ◯ observable ⊗ not-observable

6. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^{\mathrm{T}}$, the free evolution of the following discrete-time autonomous system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\mathbf{x}(k) \qquad \mathbf{x}(k) = \begin{bmatrix} 3^k & k\,3^{k-1} & \frac{k(k-1)}{2}3^{k-2} & 0 \\ 0 & 3^k & k\,3^{k-1} & 0 \\ 0 & 0 & 3^k & 0 \\ 0 & 0 & 0 & 3^k \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

7. Apply the $\mathcal{Z}$transform to the following *state* function:

$$\mathcal{Z}\left[\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)\right]$$

   and provides the expression of the transformed function $\mathbf{x}(z)$ of the state vector $\mathbf{x}(k)$ as a function of the initial state $\mathbf{x}_0$ and the transform $\mathbf{u}(z)$ of the input signal $u(k)$:

$$\mathbf{x}(z) = (z\mathbf{I} - \mathbf{A})^{-1}z\,\mathbf{x}_0 + (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\,\mathbf{u}(z)$$

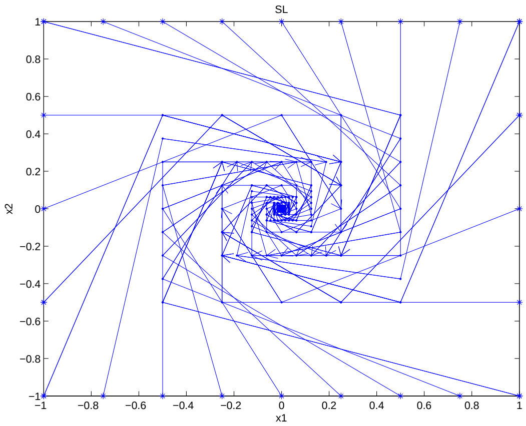
8. Considered a second order **discrete-time** dynamic system $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k)$ characterized by the two complex coniugate eigenvalues $\lambda_{1,2} = \pm 0.5\,j$, answer the following questions and draw the qualitative behavior of the state trajectories in the vicinity of the origin:

⊗ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are complex coniugate.

○ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are straight trajectories.

○ for $k \to \infty$ all the trajectories tend to flatten on one of the two eigenvectors.

⊗ for $k \to \infty$ all the trajectories tend to zero.



Which name is typically used for denoting the type of trajectories shown above:

○ Node? ⊗ Focus? ○ Saddle? | ○ Degenerate? ⊗ Stable? ○ Unstable?

9. Given the transfer function $G(z)$, write the structure of the corresponding dynamic system in the reachability canonical form denoting with $u(k)$ the input and with $y(k)$ the output:

$$G(z) = \frac{z^3 + 2z^2 + 4z + 6}{z^4 + 4z^3 + 3z^2 + 5z + 2} + 1 \qquad \left\{ \begin{array}{l} \mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -5 & -3 & -4 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k) \\ \\ y(k) = \begin{bmatrix} 6 & 4 & 2 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \end{bmatrix} u(k) \end{array} \right.$$

Without making specific calculations it can be stated that surely:

⊗ the system is reachable;

○ the system is observable;

○ the system is stable;

⊗ the system has zero relative degree;

10. Given the following continuous-time linear system:

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ -2 & 2 & 1 \\ \mathbf{0} & \mathbf{0} & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ \mathbf{0} \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{x}(t) \end{array} \right.$$

Thinking to the block structure of the systems in standard form it is possible to state that:

⊗ the system is in the standard observability form;

⊗ the system is in the reachability standard form;

○ for this system it is possibile to build an asymptotic state observer;

⊗ the system can be stabilized using a static state feedback;

Using the structural properties of the given system compute the transfer function $G(s) = \frac{Y(s)}{U(s)}$ which links the input $U(s) = \mathcal{L}[u(t)]$ to the output $Y(s) = \mathcal{L}[y(t)]$

$$G(s) = \mathbf{c}_1(s\,\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{b}_1 = \frac{1}{s-1} \qquad \text{where} \qquad \mathbf{c}_1 = [\,1\,],\;\; \mathbf{A}_{11} = [\,1\,],\;\; \mathbf{b}_1 = [\,1\,].$$

11. Given a SISO linear system of the forth order ($n = 4$), completely observable, characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$.
a) Write the structure of the matrices $\mathbf{A}_o$, $\mathbf{b}_o$ and $\mathbf{c}_o$ of the corresponding observability canonical form. Let $p(\lambda) = \lambda^4+\alpha_3\lambda^3+\alpha_2\lambda^2+\alpha_1\lambda+\alpha_0$ be the characteristic polynomial of matrix $\mathbf{A}$.

$$\mathbf{A}_o = \begin{bmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{bmatrix}, \qquad \mathbf{b}_o = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}, \qquad \mathbf{c}_o = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

b) Moreover, write the structure of matrix $\mathbf{P}$ which, together with the space transformation $\mathbf{x} = \mathbf{P}\mathbf{x}_o$, brings the system in the observability canonical form.

$$\mathbf{P} = [(\mathcal{O}_c^-)^{-1}\mathcal{O}^-]^{-1} = \left(\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 1 \\ \alpha_2 & \alpha_3 & 1 & 0 \\ \alpha_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \mathbf{cA}^2 \\ \mathbf{cA}^3 \end{bmatrix} \right)^{-1}$$

12. Write the Ackermann formula for computing the gain vector $\mathbf{l}$ of an asymptotic state observer which freely places the eigenvalues of matrix $\mathbf{A} + \mathbf{lc}$:

$$\mathbf{l} = -p(\mathbf{A})(\mathcal{O}^-)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = -p(\mathbf{A})\mathbf{q}$$

Write the structure of the desired polynomial $p(\lambda)$ and matrix $p(\mathbf{A})$ when three eigenvalues are located in $\lambda = -1$ and other two eigenvalues are located in $\lambda = -3$:

$$p(\lambda) = (\lambda + 1)^3(\lambda + 3)^2, \qquad p(\mathbf{A}) = (\mathbf{A} + \mathbf{I})^3(\mathbf{A} + 3\mathbf{I})^2$$

13. Given the following nonlinear differential equations in the state space:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -3\,x_1 \sin^2 x_2 - 2\,x_2^3 + x_3^2 \tan x_1 + u(t) \end{cases}$$

Set $[x_1 \;\; x_2 \;\; x_3]^{\mathrm{T}} = [y(t) \;\; \dot{y}(t) \;\; \ddot{y}(t)]^{\mathrm{T}}$, write the corresponding third order nonlinear differential equation which links the input $u(t)$ to the output $y(t)$:

$$\dddot{y}(t) + 3\,y(t)\sin^2 \dot{y}(t) + 2\,\dot{y}^3(t) - \ddot{y}^2(t)\tan y(t) = u(t).$$

14. Write the structure of the dual system $\mathcal{S}_D$ corresponding to a given system $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$:

$$\mathcal{S}_D = (\mathbf{A}^{\mathrm{T}},\ \mathbf{C}^{\mathrm{T}},\ \mathbf{B}^{\mathrm{T}},\ \mathbf{D}^{\mathrm{T}})$$

15. Given a system $(\mathbf{A},\ \mathbf{c})$ completely observable. The corresponding sampled system (being T the sampling period) is completely observable if and only if for each couple λ_i, λ_j of eigenvalues of $\mathbf{A}$ having the same real part, it is:

$$\mathrm{Im}(\lambda_i - \lambda_j) \neq \frac{2k\pi}{T} \qquad k = \pm 1,\ \pm 2,\ \ldots$$

16. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ which move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the solution $\mathbf{u}$ which minimizes the Euclidean norm:

$$\mathbf{u} = (\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}[\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0)]$$

17. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$, write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$ with period T:

$$\mathbf{F} = e^{\mathbf{A}T}, \qquad \mathbf{G} = \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\, d\sigma, \qquad \mathbf{H} = \mathbf{C}$$

18. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of:

a) an **open loop state estimator** and the time evolution of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ obtained starting from the initial condition $\mathbf{e}(0)$:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{e}(t) = e^{\mathbf{A}t}\,\mathbf{e}(0)$$

b) an **full order closed loop state estimator** and the time evolution of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ obtained starting from the initial condition $\mathbf{e}(0)$:

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} + \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t), \qquad \mathbf{e}(t) = e^{(\mathbf{A}+\mathbf{L}\mathbf{C})t}\,\mathbf{e}(0)$$

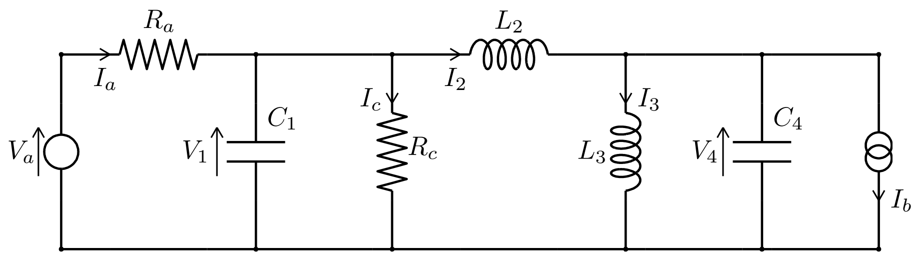
19. Write the direct Lyapunov stability criterion for continuous-time systems.

Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.

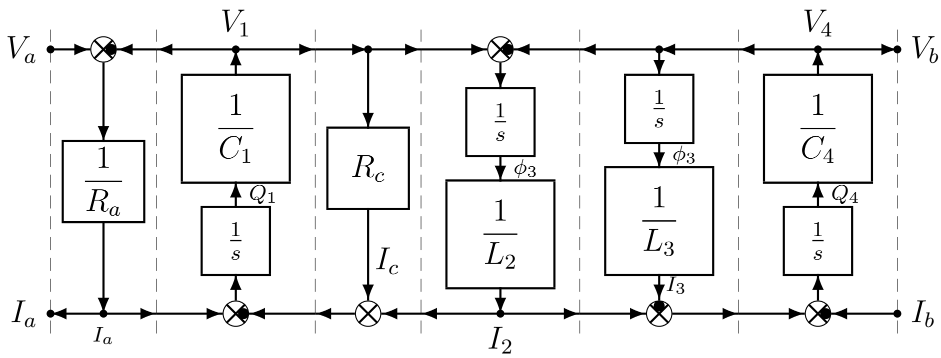
1) If in a neighborhood W *of* $\mathbf{x}_0$ *it exists a function* $V(\mathbf{x}) : W \to \mathcal{R}$ positive definite *with continuous first time-derivatives and if* $\dot{V}(\mathbf{x})$ *is* negative semidefinite, *then the point* $\mathbf{x}_0$ *is* stable *for the nonlinear system .*

2) Moreover, if $\dot{V}(\mathbf{x})$ *is* negative definite, *then the origin is* asymptotically *stable.*

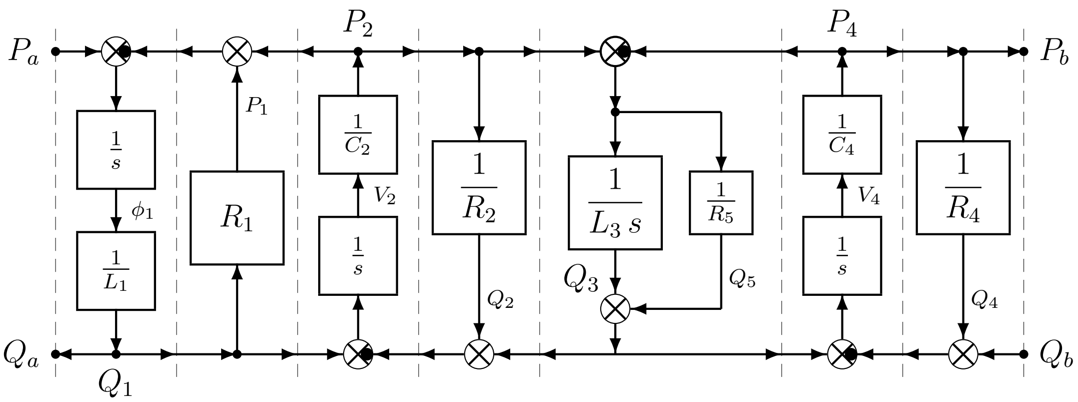
20. Consider the following electric circuit composed by the capacities C_1, C_4, the inductances L_2, L_3 and the resistances R_a and R_c. The system has two inputs: the voltage V_a and the current I_b. The outputs of the system are: the current I_a and the voltage V_b.



Write the POG model of the given electric circuit:



21. Consider the following POG scheme which describes the dynamics of an hydraulic circuit composed by the hydraulic inductances L_1, L_3, the hydraulic capacities C_2, C_4 and the hydraulic resistances R_1, R_2, R_4 and R_5:

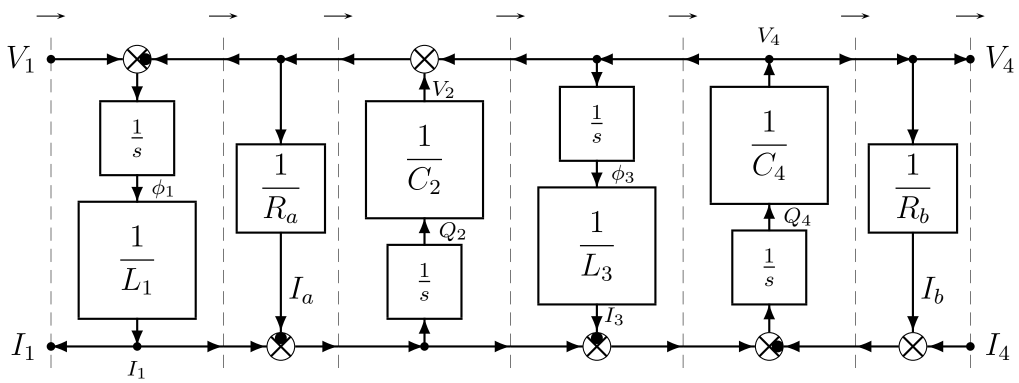


Let $\mathbf{x} = \begin{bmatrix} Q_1 & P_2 & Q_3 & P_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} P_a & Q_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} Q_a & P_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{Q}_1 \\ \dot{P}_2 \\ \dot{Q}_3 \\ \dot{P}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_1 & -1 & 0 & 0 \\ 1 & -\frac{1}{R_2}-\frac{1}{R_5} & -1 & \frac{1}{R_5} \\ 0 & 1 & 0 & -1 \\ 0 & \frac{1}{R_5} & 1 & -\frac{1}{R_4}-\frac{1}{R_5} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} Q_1 \\ P_2 \\ Q_3 \\ P_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} Q_a \\ P_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

22. Add the **minus signs** to the following POG scheme in order to obtain a POG scheme correctly defined according the given positive directions of the power flows:



23. Which of the following functions $V(x_1, x_2)$ are positive definite in the vicinity of the origin?

⊗ $V(x_1,x_2) = x_1^2 + x_1^3 + x_2^2 + x_2^3$;

○ $V(x_1,x_2) = x_1^4 \sin(x_2) + x_2^2 \cos^2(x_1)$;

⊗ $V(x_1,x_2) = x_1^2(1-x_1^3) + x_2^2(1-x_2^3)$;

⊗ $V(x_1,x_2) = x_1^2 \sin^2(x_2) + x_2^4 \cos(x_1)$;

24. Given the following nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x})$, continuous-time and autonomous:

$$\begin{cases} \dot{x}_1 &= -x_1^3 + \alpha\, x_2 + x_3^6 \\ \dot{x}_2 &= -\alpha\, x_1 - \, x_2^3 \\ \dot{x}_3 &= 2\beta\, x_3 - x_3^5 \end{cases}$$

It is easy to verify that the origin $\mathbf{x}_0 = (0,\, 0,\, 0) = \mathbf{0}$ is an equilibrium point for the system.

a) Compute, as a function of parameters α and β, the Jacobian $\mathbf{A}(\mathbf{x})$ of the nonlinear system:

The Jacobian $\mathbf{A}(\mathbf{x})$ has the following structure:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} -3x_1^2 & \alpha & 6x_3^5 \\ -\alpha & -3x_2^2 & 0 \\ 0 & 0 & 2\beta - 5x_3^4 \end{bmatrix}$$

b) Compute, as a function of α and β, the matrix $\mathbf{A}_0$ of the linearized system at point $\mathbf{x}_0 = \mathbf{0}$:

The matrix $\mathbf{A}_0$ of the linearized system has the following structure:

$$\mathbf{A}_0 = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 2\beta \end{bmatrix}$$

c) Study, for varying α and β, the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = \mathbf{0}$ using the reduced Lyapunov criterion:

The eigenvalues of matrix $\mathbf{A}_0$ are:

$$\lambda_{1,2} = \pm j\alpha. \qquad\qquad \lambda_3 = 2\beta,$$

Using the reduced Lyapunov criterion it can be stated that: 1) for $\beta > 0$ and $\forall\alpha$ the equilibrium point $\mathbf{x}_0 = \mathbf{0}$ of the nonlinear system is unstable; 2) for $\beta \leq 0$ and $\forall\alpha$ the criterion cannot be used.

d) For $\beta = 0$, study for varying parameter α the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = \mathbf{0}$ using the "direct" Lyapunov criterion and the function: $V(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2$. Eventually, use the La Salle - Krasowskii criterion.

In the neighborhood of point $\mathbf{x}_0 = \mathbf{0}$ the function $V(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2$ is positive definite. For $\beta = 0$, the function $\dot{V}(\mathbf{x})$ computed along the system's trajectories is:

$$\begin{aligned} \dot{V}(\mathbf{x}) &= 2x_1(-x_1^3 + \alpha\, x_2 + x_3^6) + 2x_2(-\alpha\, x_1 - \, x_2^3) + 2x_3(-x_3^5) \\ &= -2x_1^4 - 2\, x_2^4 - 2x_3^6(1 - x_1) < 0 \end{aligned}$$

In point $\mathbf{x}_0 = \mathbf{0}$ the function $\dot{V}(\mathbf{x})$ is negative definite and therefore, using the "direct" Lyapunov criterion, it can be stated that in the neighborhood of point $\mathbf{x}_0 = \mathbf{0}$ the nonlinear system is asymptotically stable for all the values of α.

25. Compute the 2 equilibrium points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ of the following *discrete-time* nonlinear system:

$$\begin{cases} x_1(k+1) &= 2\, x_2(k) \\ x_2(k+1) &= x_1(k) + x_2(k)(3 - x_1(k)) \end{cases} \quad \underset{\mathbf{x}(k+1)=\mathbf{x}(k)}{\Rightarrow} \quad \begin{cases} \tilde{\mathbf{x}}_1 &= (\;0,\;\; 0\;) \\ \tilde{\mathbf{x}}_2 &= (\;4,\;\; 2\;) \end{cases}$$