System and Control Theory
Test of February 11, 2016
Questions and Exercises

Name:	
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1. Write the discrete time behavior of the output function $\mathbf{y}(k)$, solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ and the static equation $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$ starting from the initial condition $\mathbf{x}(h)$ at time h:

$$\mathbf{y}(k) =$$

2. Write the closed form solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$ at time $t = t_0$:

$$\mathbf{x}(t) =$$

3. Give the meaning of the symbol $\mathcal{E}^+(t_0, t_1, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$:

   ...

4. Give the meaning, for discrete linear systems, of the symbol $\mathcal{X}^+(k)$:

   $\mathcal{X}^+(k)$ is ...

5. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \quad \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \quad \end{bmatrix},$$

   The system is: ◯ reachable? ◯ not-reachable? ◯ observable? ◯ not-observable?

   Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \mathrm{Im}\,[\mathcal{B}_R] = \mathrm{Im}\begin{bmatrix} \quad \end{bmatrix}, \qquad \mathcal{E}^- = \mathrm{Im}\,[\mathcal{B}_O] = \mathrm{Im}\begin{bmatrix} \quad \end{bmatrix}.$$

6. Compute, as function of the initial condition $\mathbf{x}_0 = [x_{10},\ x_{20},\ x_{30},\ x_{40}]^{\mathrm{T}}$, the free evolution of the following continuous-time autonomous system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{x}(t), \qquad \mathbf{x}(t) = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix}$$

7. Write the formula for computing the state transition matrix $\mathbf{A}^k$ of a discrete-time linear system using the $\mathcal{Z}$-transform:

$$\mathbf{A}^k =$$

8. Apply the Laplace transform to the following *state* function:

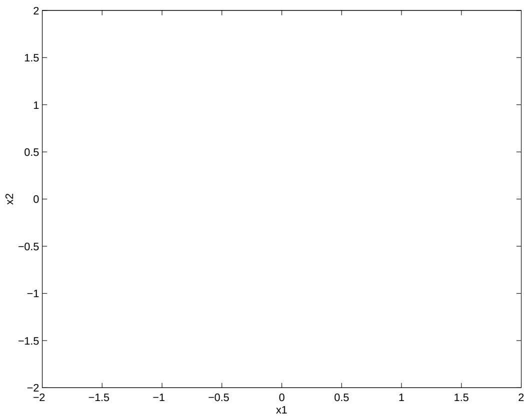
$$\mathcal{L}\left[\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right]$$

and provides the expression of the transformed $\mathbf{x}(s)$ of the state vector $\mathbf{x}(t)$ as a function of the initial state $\mathbf{x}_0$ and the transformed $\mathbf{u}(s)$ of the input signal $\mathbf{u}(t)$:

$$\mathbf{x}(s) =$$

9. Considered a second order **discrete-time** dynamic system $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k)$ characterized by two real eigenvalues $\lambda_1 = 0.7$, $\lambda_2 = 0.9$, answer the following questions and draw the qualitative behavior of the state trajectories in the vicinity of the origin:

- ◯ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are real and different.
- ◯ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are straight trajectories.
- ◯ for $k \to \infty$ all the trajectories tend to flatten on one of the two eigenvectors.
- ◯ for $k \to \infty$ all the trajectories tend to zero.



Which name is typically used for denoting the type of trajectories shown above:

◯ Node? ◯ Focus? ◯ Saddle? | ◯ Degenerate? ◯ Stable? ◯ Unstable?

10. Given the following dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t)$, $\mathbf{y}(t) = \mathbf{c}\,\mathbf{x}(t) + d\,u(t)$, compute the function $G(s) = \frac{Y(s)}{U(s)}$ which links the input $U(s) = \mathcal{L}[u(t)]$ to the output $Y(s) = \mathcal{L}[y(t)]$:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7 & -3 & -5 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} 2 & 6 & 0 & 4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \end{bmatrix} u(t) \end{cases} \qquad G(s) =$$

Without making specific calculations it can be stated that surely:

- ◯ the system is reachable;
- ◯ the system is controllable;
- ◯ the system is observable;
- ◯ the system is constructable;

11. Consider the following discrete control action:

$$\mathbf{u}(k) = (\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}[\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0)]$$

Describe the type of trajectory obtained in the state space if you are using the control action shown above:

...

12. An "closed loop" state estimator can be used
   - ◯ if the system is observable;
   - ◯ if the system is asymptotically stable;
   - ◯ if the unstable part of the system is observable;
   - ◯ even if the not-observable part of the system is unstable;

13. A system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ can be "stabilized" using a static state feedback
   - ◯ even if the system has all its eigenvalues unstable;
   - ◯ if and only if the system is completely reachable;
   - ◯ if and only if the unstable part of the system is reachable;
   - ◯ if and only if the reachable part of the system is unstable;

14. The Ackermann formula for computing the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of the feedback system can be used
   - ◯ only for linear systems;
   - ◯ also for unstable systems;
   - ◯ also for not-reachable systems;
   - ◯ also for systems with more inputs;

   Provide the explicit descriptions of the Ackermann formula and the desired polynomial $p(\lambda)$ which places all the n eigenvalues of the system in the same point λ_0 imposing a settling time $T_a = 0.6$:

   $\mathbf{k}^{\mathrm{T}} =$ $\qquad\qquad$ $p(\lambda) =$ $\qquad\qquad$ $\lambda_0 =$

15. Write the block matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$ of a system in the reachability standard form:

   $$\overline{\mathbf{A}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \qquad \overline{\mathbf{B}} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}, \qquad \overline{\mathbf{C}} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

16. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of a **full order closed loop** state estimator and the time evolution of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ obtained starting from the initial condition $\mathbf{e}(0)$:

   $\dot{\hat{\mathbf{x}}}(t) =$ $\qquad\qquad$ $\mathbf{e}(t) =$ $\qquad\qquad$.

17. Write the necessary and sufficient condition which guarantees the controllability in k steps of the discrete linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$:

18. Given the following nonlinear **difference equation**:

   $$y(k+3) + 2\,y(k)\sin y(k+2) + 3\,y(k+2)y(k+1) + 7\,\cos y(k) = u(k)$$

   and chosen $\mathbf{x} = [x_1(k)\;\; x_2(k)\;\; x_3(k)]^{\mathrm{T}} = [y(k)\;\; y(k+1)\;\; y(k+2)]^{\mathrm{T}}$ as state vector, rewrite the nonlinear difference equation in the state space:

   $$\begin{cases} x_1(k+1) & = \\ x_2(k+1) & = \\ x_3(k+1) & = \end{cases}$$

19. Given a SISO linear system, completely reachable, characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$. Write the structure of matrix $\mathbf{T}$ which (using state space transformation $\mathbf{x} = \mathbf{T}\mathbf{x}_c$) brings the system in controllability canonical form.

$\mathbf{T} =$

20. Let $\mathcal{S}_D$ be the dual system of the **continuous-time** system $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$:

- ◯ If $\mathcal{S}$ is controllable $\Rightarrow$ $\mathcal{S}_D$ is constructable;
- ◯ If $\mathcal{S}$ is reachable $\Rightarrow$ $\mathcal{S}_D$ is observable;
- ◯ If $\mathcal{S}$ is observable $\Rightarrow$ $\mathcal{S}_D$ is constructable;
- ◯ If $\mathcal{S}$ is constructable $\Rightarrow$ $\mathcal{S}_D$ is controllable;

21. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$, write the expression of the matrixs $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$ with period T:

$\mathbf{F} =$ $\mathbf{G} =$ $\mathbf{H} =$

22. For the linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,u(t)$, the input function $u(\tau)$, with $\tau \in [0,\ t_1]$, which moves the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(t_1)$:

- ◯ exists if $\mathbf{x}(t_1) \in \mathcal{X}^+$;
- ◯ exists if $\mathbf{x}(t_1) - e^{\mathbf{A}t_1}\mathbf{x}(0) \in \mathcal{X}^+$;
- ◯ exists if the system is reachable;
- ◯ exists if $\mathbf{x}(t_1) - \mathbf{A}^{t_1}\mathbf{x}(0) \in \mathcal{X}^+$;

23. Write the Lyapunov instability criterion for discrete-time nonlinear systems.
Consider the nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k),\ \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$*. If:*

1) ...
2) ...
3) ...

then ...

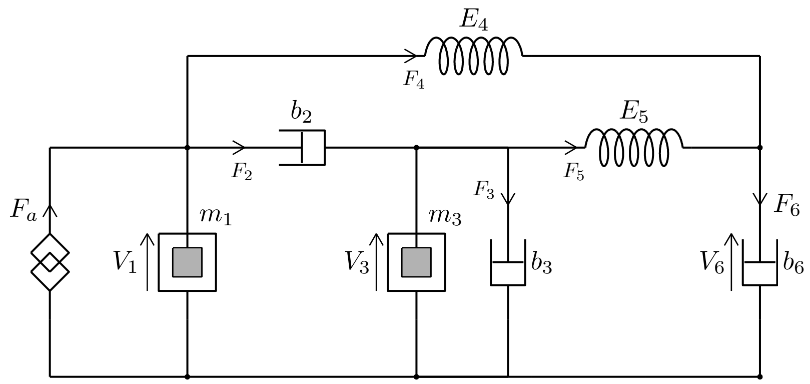
24. *Given a discrete-time nonlinear system* $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$*. The equilibrium points of this system can be obtained:*

- ◯ *computing the points* $\mathbf{x}_e$ *for which it is* $\mathbf{x}_e = \mathbf{f}(\mathbf{x}_e)$.
- ◯ *computing the points* $\mathbf{x}_e$ *for which it is* $\mathbf{f}(\mathbf{x}_e) = 0$.
- ◯ *computing the points* $\mathbf{x}_e(k)$ *of the state space for which it is* $\mathbf{x}_e(k+1) = \mathbf{x}_e(k)$.

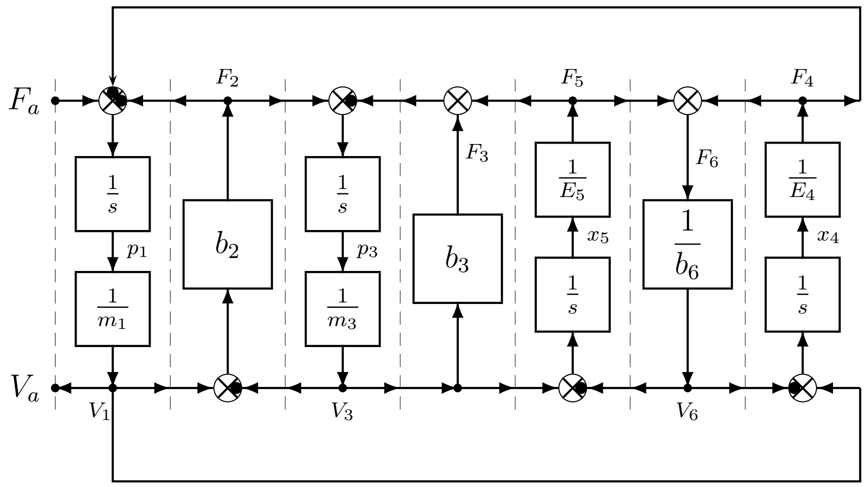
25. *Write the energy variables* q_1, q_2 *and the output power variables* v_1 *and* v_2 *of the dynamic elements* $\mathcal{D}_1$ *and* $\mathcal{D}_1$ *that characterize the energetic domains:*

	Electrical	**Mech. Trans.**	**Mech. Rot.**	**Hydraulic**
$\mathcal{D}_1$	C *Capacity*	M *Mass*	J *Inertia*	C_I *Hydr. Capacity*
q_1				
v_1				
$\mathcal{D}_2$	L *Inductance*	E *Spring*	E *Tors. Spring*	L_I *Hydr. Inductance*
q_2				
v_2				

26. *Consider the following mechanical system composed by masses m_1, m_3, elasticities E_4, E_5 and dampers b_2, b_3 and b_6. On the system acts only one input: the force F_a. The outputs of the system are: velocity $V_a = V_1$ and the force $F_b = F_4 + F_5$.*



The POG model of the given mechanical system has the following structure:



Let $\mathbf{x} = \begin{bmatrix} V_1 & V_3 & F_5 & F_4 \end{bmatrix}^T$ *be the state vector,* $\mathbf{u} = F_a$ *the input vector and* $\mathbf{y} = \begin{bmatrix} V_a & F_6 \end{bmatrix}^T$ *the output vector. Write the corresponding dynamic system* $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ *and* $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ *in the state space:*

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{V}_3 \\ \dot{F}_5 \\ \dot{F}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} V_1 \\ V_3 \\ F_5 \\ F_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} F_a \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ F_6 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} F_a \end{bmatrix}}_{\mathbf{u}}$$

27. *Which of the following functions $V(x_1, x_2)$ are positive definite in the neighborhood* **of point (1, 1)***:*

- ◯ $V(x_1, x_2) = |x_1 - 1|x_2 + |x_2 - 1|x_1$;
- ◯ $V(x_1, x_2) = |x_1 - 1| + |x_2 - 1|$;
- ◯ $V(x_1, x_2) = (|x_1| - 1)x_2 + (|x_2| - 1)x_1$;
- ◯ $V(x_1, x_2) = (|x_1| - 1) + (|x_2| - 1)$;

28. *Given the following continuous-time nonlinear system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$*:*

$$\begin{cases} \dot{x}_1 &= x_2^4 - x_1^3 - 2x_1 \\ \dot{x}_2 &= \alpha x_2 - x_1^3 x_2^3 + u \end{cases}$$

Set $u = 0$*, it is easy to verify that the origin* $\mathbf{x}_1 = (0,\ 0)$ *is an equilibrium point for the system.*

a) Compute the matrix $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x},u)}{\partial \mathbf{x}}$ *and the vector* $\mathbf{b} = \frac{\partial \mathbf{f}(\mathbf{x},u)}{\partial u}$ *of the nonlinear system:*

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

b) Compute matrix $\mathbf{A}_1$ *and vector* $\mathbf{b}_1$ *of the linearized system at the point* $\mathbf{x}_1 = (0, 0)$*,* $u = 0$*:*

$$\mathbf{A}_1 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \qquad \mathbf{b}_1 = \begin{bmatrix} \quad \\ \quad \end{bmatrix},$$

c) Study, for varying parameter α*, the stability of the nonlinear system in the neighborhood of point* $\mathbf{x}_1 = (0,\ 0)$*,* $u = 0$ *using the reduced Lyapunov criterion:*

d) For $\alpha = 0$ *and* $u = 0$*, study the stability of the nonlinear system in the vicinity of the origin* $\mathbf{x}_1 = (0, 0)$ *using the "direct" Lyapunov criterion and the function:* $V(\mathbf{x}) = x_1^4 + 2x_2^2$*. Eventually, use the La Salle - Krasowskii criterion.*

e) Set $\alpha = 0$ *and* $u = k_1 x_1 + k_2 x_2$*, determine for which values of parameters* k_1 *and* k_2 *the linearized system is asymptotically stable in the neighborhood of point* $\mathbf{x}_1 = (0, 0)$*.*