System and Control Theory
Test of February 11, 2016
Questions and Exercises

Name:	
Nr. Mat.	
Signature:	

1. Write the discrete time behavior of the output function $\mathbf{y}(k)$, solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ and the static equation $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$ starting from the initial condition $\mathbf{x}(h)$ at time h:

$$\mathbf{y}(k) = \mathbf{C}\,\mathbf{A}^{k-h}\mathbf{x}(h) + \mathbf{C}\sum_{j=h}^{k-1}\mathbf{A}^{k-j-1}\mathbf{B}\mathbf{u}(j) + \mathbf{D}\mathbf{u}(k)$$

2. Write the closed form solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$ at time $t = t_0$:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

3. Give the meaning of the symbol $\mathcal{E}^+(t_0, t_1, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$:

It is the set of the final states $\mathbf{x}(t_1)$ *compatible with the input and output functions* $\mathbf{u}(\cdot)$ *and* $\mathbf{y}(\cdot)$ *in the time interval* $[t_0, t_1]$.

4. Give the meaning, for discrete linear systems, of the symbol $\mathcal{X}^+(k)$:

$\mathcal{X}^+(k)$ *is the set of the states* reachable in k steps, *that is the states that can be reached from the origin* $\mathbf{x}(0) = 0$ *in* k *steps considering all the possible input sequences* $\mathbf{u}(\tau)$.

5. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix},$$

The system is: ○ reachable? ⊗ not-reachable? ○ observable? ⊗ not-observable?

Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \mathrm{Im}\,[\mathcal{B}_R] = \mathrm{Im} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathcal{E}^- = \mathrm{Im}\,[\mathcal{B}_O] = \mathrm{Im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

6. Compute, as function of the initial condition $\mathbf{x}_0 = [x_{10},\ x_{20},\ x_{30},\ x_{40}]^{\mathrm{T}}$, the free evolution of the following continuous-time autonomous system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{x}(t), \qquad \mathbf{x}(t) = \begin{bmatrix} e^t\cos 2t & e^t \sin 2t & 0 & 0 \\ -e^t \sin 2t & e^t \cos 2t & 0 & 0 \\ 0 & 0 & e^{-t} & te^{-t} \\ 0 & 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix}$$

7. Write the formula for computing the state transition matrix $\mathbf{A}^k$ of a discrete-time linear system using the $\mathcal{Z}$-transform:

$$\mathbf{A}^k = \mathcal{Z}^{-1}[z(z\mathbf{I} - \mathbf{A})^{-1}]$$

8. Apply the Laplace transform to the following *state* function:

$$\mathcal{L}\left[\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right]$$

and provides the expression of the transformed $\mathbf{x}(s)$ of the state vector $\mathbf{x}(t)$ as a function of the initial state $\mathbf{x}_0$ and the transformed $\mathbf{u}(s)$ of the input signal $\mathbf{u}(t)$:

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s)$$

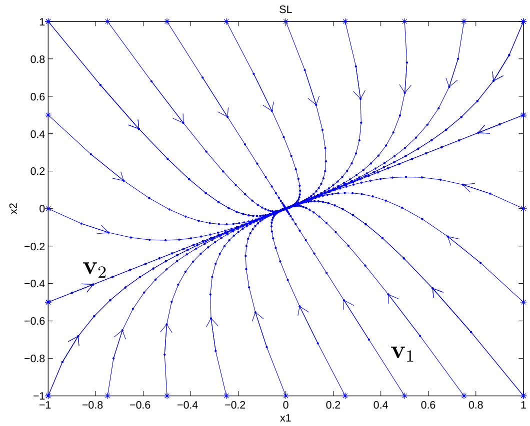
9. Considered a second order **discrete-time** dynamic system $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k)$ characterized by two real eigenvalues $\lambda_1 = 0.7$, $\lambda_2 = 0.9$, answer the following questions and draw the qualitative behavior of the state trajectories in the vicinity of the origin:

⊗ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are real and different.

⊗ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are straight trajectories.

⊗ for $k \to \infty$ all the trajectories tend to flatten on one of the two eigenvectors.

⊗ for $k \to \infty$ all the trajectories tend to zero.



Which name is typically used for denoting the type of trajectories shown above:

⊗ Node? ◯ Focus? ◯ Saddle? | ◯ Degenerate? ⊗ Stable? ◯ Unstable?

10. Given the following dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t)$, $\mathbf{y}(t) = \mathbf{c}\,\mathbf{x}(t) + d\,u(t)$, compute the function $G(s) = \frac{Y(s)}{U(s)}$ which links the input $U(s) = \mathcal{L}[u(t)]$ to the output $Y(s) = \mathcal{L}[y(t)]$:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7 & -3 & -5 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & 6 & 0 & 4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \end{bmatrix} u(t) \end{cases} \qquad G(s) = \frac{4s^3 + 6s + 2}{s^4 + s^3 + 5s^2 + 3s + 7} + 1$$

Without making specific calculations it can be stated that surely:

⊗ the system is reachable; ⊗ the system is controllable;

◯ the system is observable; ◯ the system is constructable;

11. Consider the following discrete control action:

$$\mathbf{u}(k) = (\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}[\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0)]$$

Describe the type of trajectory obtained in the state space if you are using the control action shown above:

$\mathbf{u}(k)$ *is the control action with minimum Euclidian norm that moves the system from the initial state* $\mathbf{x}(0)$ *to the final state* $\mathbf{x}(k)$ *in the time interval* $[0,\ k]$.

12. An "closed loop" state estimator can be used

⊗ if the system is observable;

- ⊗ if the system is asymptotically stable;
- ⊗ if the unstable part of the system is observable;
- ○ even if the not-observable part of the system is unstable;

13. A system ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$) can be "stabilized" using a static state feedback

- ⊗ even if the system has all its eigenvalues unstable;
- ○ if and only if the system is completely reachable;
- ⊗ if and only if the unstable part of the system is reachable;
- ○ if and only if the reachable part of the system is unstable;

14. The Ackermann formula for computing the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of the feedback system can be used

- ⊗ only for linear systems;
- ⊗ also for unstable systems;
- ○ also for not-reachable systems;
- ○ also for systems with more inputs;

Provide the explicit descriptions of the Ackermann formula and the desired polynomial $p(\lambda)$ which places all the n eigenvalues of the system in the same point λ_0 imposing a settling time $T_a = 0.6$:

$$\mathbf{k}^{\mathrm{T}} = -\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} (\mathcal{R}^{+})^{-1} p(\mathbf{A}), \qquad p(\lambda) = (\lambda - \lambda_0)^n, \qquad \lambda_0 = -\frac{3}{T_a} = -5$$

15. Write the block matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$ of a system in the reachability standard form:

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ 0 & \mathbf{A}_{2,2} \end{bmatrix} \qquad \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}$$

16. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of a **full order closed loop** state estimator and the time evolution of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ obtained starting from the initial condition $\mathbf{e}(0)$:

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} + \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t), \qquad \mathbf{e}(t) = e^{(\mathbf{A}+\mathbf{L}\mathbf{C})t}\,\mathbf{e}(0).$$

17. Write the necessary and sufficient condition which guarantees the controllability in k steps of the discrete linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$:

$$\mathrm{Im}\mathbf{A}^k \subseteq \mathrm{Im}[\mathbf{B}, \mathbf{A}\mathbf{B}, \dots, \mathbf{A}^{k-1}\mathbf{B}] = \mathcal{X}^{+}(k)$$

18. Given the following nonlinear **difference equation**:

$$y(k+3) + 2\,y(k)\sin y(k+2) + 3\,y(k+2)y(k+1) + 7\,\cos y(k) = u(k)$$

and chosen $\mathbf{x} = [x_1(k)\;\; x_2(k)\;\; x_3(k)]^{\mathrm{T}} = [y(k)\;\; y(k+1)\;\; y(k+2)]^{\mathrm{T}}$ as state vector, rewrite the nonlinear difference equation in the state space:

$$\begin{cases} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ x_3(k+1) &= -2\,x_1(k)\sin x_3(k) - 3\,x_3(k)\,x_2(k) - 7\,\cos x_1(k) + u(k) \end{cases}$$

19. Given a SISO linear system, completely reachable, characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$. Write the structure of matrix $\mathbf{T}$ which (using state space transformation $\mathbf{x} = \mathbf{T}\mathbf{x}_c$) brings the system in controllability canonical form.

$$\mathbf{T} = \mathcal{R}^+(\mathcal{R}_c^+)^{-1} = \left[\begin{array}{ccccc} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \dots & \mathbf{A}^{n-1}\mathbf{b} \end{array}\right] \left[\begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \dots & \dots & 1 & 0 \\ \alpha_3 & \dots & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \dots & \dots & 0 & 0 \\ 1 & 0 & \dots & \dots & 0 & 0 \end{array}\right]$$

20. Let $\mathcal{S}_D$ be the dual system of the **continuous-time** system $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$:

⊗ If $\mathcal{S}$ is controllable $\Rightarrow \mathcal{S}_D$ is constructable;
○ If $\mathcal{S}$ is observable $\Rightarrow \mathcal{S}_D$ is constructable;
⊗ If $\mathcal{S}$ is reachable $\Rightarrow \mathcal{S}_D$ is observable;
⊗ If $\mathcal{S}$ is constructable $\Rightarrow \mathcal{S}_D$ is controllable;

21. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$, write the expression of the matrixs $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$ with period T:

$$\mathbf{F} = e^{\mathbf{A}T}, \qquad \mathbf{G} = \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\, d\sigma, \qquad \mathbf{H} = \mathbf{C}$$

22. For the linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,u(t)$, the input function $u(\tau)$, with $\tau \in [0,\ t_1]$, which moves the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(t_1)$:

○ exists if $\mathbf{x}(t_1) \in \mathcal{X}^+$;
⊗ exists if the system is reachable;
⊗ exists if $\mathbf{x}(t_1) - e^{\mathbf{A}t_1}\mathbf{x}(0) \in \mathcal{X}^+$;
○ exists if $\mathbf{x}(t_1) - \mathbf{A}^{t_1}\mathbf{x}(0) \in \mathcal{X}^+$;

23. Write the Lyapunov instability criterion for discrete-time nonlinear systems.
Consider the nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k),\ \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$*. If:*

1) in a neighborhood W of $\mathbf{x}_0$ it exists a continuous function $V(\mathbf{x}) : W \to \mathcal{R}$ which is zero in $\mathbf{x}_0$;
2) the point $\mathbf{x}_0$ is an accumulation point for the set of points $\mathbf{x} \in W$ in which it is $V(\mathbf{x}) > 0$;
3) $\Delta V(\mathbf{x})$ is *positive definite* in W;

then $\mathbf{x}_0$ is an *unstable* equilibrium point for the given nonlinear system.

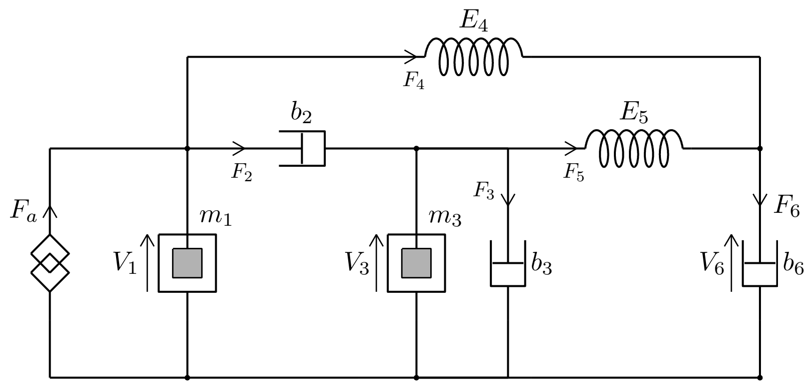
24. Given a discrete-time nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$. The equilibrium points of this system can be obtained:

⊗ computing the points $\mathbf{x}_e$ for which it is $\mathbf{x}_e = \mathbf{f}(\mathbf{x}_e)$.
○ computing the points $\mathbf{x}_e$ for which it is $\mathbf{f}(\mathbf{x}_e) = 0$.
⊗ computing the points $\mathbf{x}_e(k)$ of the state space for which it is $\mathbf{x}_e(k+1) = \mathbf{x}_e(k)$.

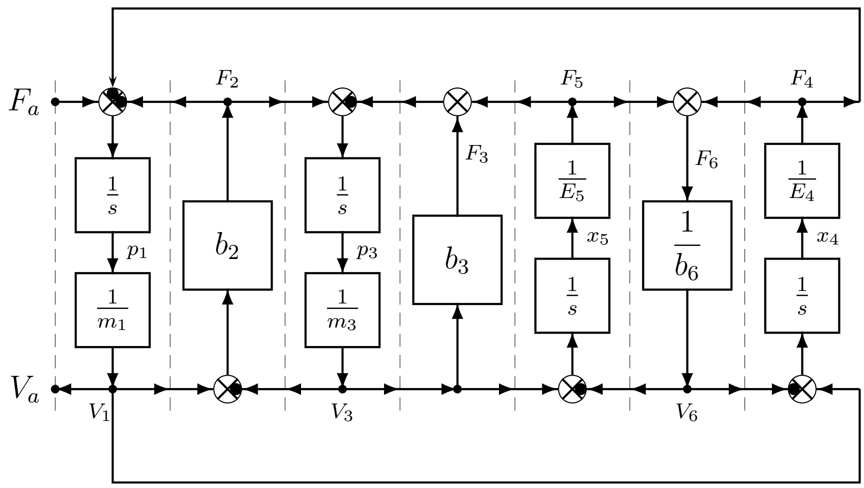
25. Write the energy variables q_1, q_2 and the output power variables v_1 and v_2 of the dynamic elements $\mathcal{D}_1$ and $\mathcal{D}_1$ that characterize the energetic domains:

	Electrical		**Mech. Trans.**		**Mech. Rot.**		**Hydraulic**	
$\mathcal{D}_1$	C	Capacity	M	Mass	J	Inertia	C_I	Hydr. Capacity
q_1	Q	Charge	p	Momentum	p	Ang. Momentum	V	Volume
v_1	V	Voltage	v	Velocity	ω	Ang. Velocity	P	Pressure
$\mathcal{D}_2$	L	Inductance	E	Spring	E	Tors. Spring	L_I	Hydr. Inductance
q_2	ϕ	Flux	x	Displacement	θ	Ang. Displacement	ϕ_I	Hudr. Flux
v_2	I	Current	F	Force	τ	Torque	Q	Volume flow rate

26. Consider the following mechanical system composed by masses m_1, m_3, elasticities E_4, E_5 and dampers b_2, b_3 and b_6. On the system acts only one input: the force F_a. The outputs of the system are: velocity $V_a = V_1$ and the force $F_b = F_4 + F_5$.



The POG model of the given mechanical system has the following structure:



Let $\mathbf{x} = \begin{bmatrix} V_1 & V_3 & F_5 & F_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = F_a$ the input vector and $\mathbf{y} = \begin{bmatrix} V_a & F_6 \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_3 & 0 & 0 \\ 0 & 0 & E_5 & 0 \\ 0 & 0 & 0 & E_4 \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{V}_3 \\ \dot{F}_5 \\ \dot{F}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -b_2 & b_2 & 0 & -1 \\ b_2 & -b_2 - b_3 & -1 & 0 \\ 0 & 1 & -\frac{1}{b_6} & -\frac{1}{b_6} \\ 1 & 0 & -\frac{1}{b_6} & -\frac{1}{b_6} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} V_1 \\ V_3 \\ F_5 \\ F_4 \end{bmatrix}}_{\mathbf{X}} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} F_a \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ F_6 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} F_a \end{bmatrix}}_{\mathbf{u}}$$

27. Which of the following functions $V(x_1, x_2)$ are positive definite in the neighborhood **of point (1, 1)**:

⊗ $V(x_1, x_2) = |x_1 - 1|x_2 + |x_2 - 1|x_1$;

⊗ $V(x_1, x_2) = |x_1 - 1| + |x_2 - 1|$;

○ $V(x_1, x_2) = (|x_1| - 1)x_2 + (|x_2| - 1)x_1$;

○ $V(x_1, x_2) = (|x_1| - 1) + (|x_2| - 1)$;

28. Given the following continuous-time nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$:

$$\begin{cases} \dot{x}_1 &= x_2^4 - x_1^3 - 2x_1 \\ \dot{x}_2 &= \alpha x_2 - x_1^3 x_2^3 + u \end{cases}$$

Set $u = 0$, it is easy to verify that the origin $\mathbf{x}_1 = (0,\, 0)$ is an equilibrium point for the system.

a) Compute the matrix $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x},u)}{\partial \mathbf{x}}$ and the vector $\mathbf{b} = \frac{\partial \mathbf{f}(\mathbf{x},u)}{\partial u}$ of the nonlinear system:

The matrix $\mathbf{A}(\mathbf{x})$ and the vector $\mathbf{b}$ hanno the following structure:

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} -3x_1^2 - 2 & 4x_2^3 \\ -3x_1^2 x_2^3 & \alpha - 3x_1^3 x_2^2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

b) Compute matrix $\mathbf{A}_1$ and vector $\mathbf{b}_1$ of the linearized system at the point $\mathbf{x}_1 = (0, 0)$, $u = 0$:

The matrix $\mathbf{A}_1$ and the vector $\mathbf{b}_1$ of the linearized system have the following structure:

$$\mathbf{A}_1 = \begin{bmatrix} -2 & 0 \\ 0 & \alpha \end{bmatrix}, \qquad \mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

c) Study, for varying parameter α, the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_1 = (0,\, 0)$, $u = 0$ using the reduced Lyapunov criterion:

The characteristic polynomial and the eigenvalues of matrix $\mathbf{A}_1$ are:

$$\Delta_{\mathbf{A}_1}(s) = (s + 2)(s - \alpha) = 0 \qquad \rightarrow \qquad s = -1, \quad s = \alpha$$

Using the reduced Lyapunov criterion it can be stated that for $\alpha > 0$ the equilibrium point $\mathbf{x}_1 = (0, 0)$ of the nonlinear system is unstable, while for $\alpha < 0$ the equilibrium point $\mathbf{x}_1$ is asymptotically stable. For $\alpha = 0$ the criterion cannot be used.

d) For $\alpha = 0$ and $u = 0$, study the stability of the nonlinear system in the vicinity of the origin $\mathbf{x}_1 = (0, 0)$ using the "direct" Lyapunov criterion and the function: $V(\mathbf{x}) = x_1^4 + 2x_2^2$. Eventually, use the La Salle - Krasowskii criterion.

In the neighbourhood of the origin the function $V(\mathbf{x}) = x_1^4 + 2x_2^2$ is surely positive definite. The time derivative of function $V(\mathbf{x})$ along the trajectories of the system for $\alpha = 0$ and $u = 0$ is the following:

$$\dot{V} = 4x_1^3(x_2^4 - x_1^3 - 2x_1) + 4x_2(-x_1^3 x_2^3) = -4x_1^6 - 8x_1^4 \leq 0$$

The function $\dot{V}$ is negative semidefinite and therefore, using the "direct" Lyapunov criterion, it can be stated that in the vicinity of the origin the nonlinear system is simply stable. The set $\mathcal{N} = \{(0,\, x_2), x_2 \in R\}$ of the points that nullify the $\dot{V}$ does not contain perturbed trajectories of the system and therefore, using the La Salle - Krasowskii criterion, it can be stated that for $\alpha = 0$ the nonlinear system is asymptotically stable in the vicinity of the origin.

e) Set $\alpha = 0$ and $u = k_1 x_1 + k_2 x_2$, determine for which values of parameters k_1 and k_2 the linearized system is asymptotically stable in the neighborhood of point $\mathbf{x}_1 = (0, 0)$.

Set $\mathbf{k}^{\mathrm{T}} = [k_1\ k_2]$, the matrix $\mathbf{A}_k$ of the linearized system in the vicinity of the origin has the following form:

$$\mathbf{A}_k = \mathbf{A}_1 + \mathbf{b}_1 \mathbf{k}^{\mathrm{T}} = \begin{bmatrix} -2 & 0 \\ k_1 & k_2 \end{bmatrix}$$

The linearized system is asymptotically stable for $k_2 < 0$ and $\forall k_1$.