System and Control Theory
Test of Genuary 11, 2016
Questions and Exercises

Name:	
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1. Write the general solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ starting from the initial condition $\mathbf{x}(0)$ at time $h = 0$:

$$\mathbf{x}(k) = \ldots$$

2. Write the discrete time behavior of the output function $\mathbf{y}(t)$, solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and the static equation $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(0)$ at time $t_0 = 0$:

$$\mathbf{y}(t) =$$

3. A dynamic system characterized by the state function $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k))$

   ◯ is a dynamic system; ◯ is a linear system;
   ◯ is a continuous-time system; ◯ is a time-invariant system;

4. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \quad \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \quad \end{bmatrix}$$

   The system is: ◯ reachable? ◯ not-reachable? ◯ observable? ◯ not-observable?

   Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \mathrm{Im}\,[\mathcal{B}_R] = \mathrm{Im}\begin{bmatrix} \quad \end{bmatrix}, \qquad \mathcal{E}^- = \mathrm{Im}\,[\mathcal{B}_O] = \mathrm{Im}\begin{bmatrix} \quad \end{bmatrix}.$$

5. Given a discrete-time linear system: $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ and $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$. Write the expression of the output *free evolution* $\mathbf{y}(k)$ of the system starting from the initial condition $\mathbf{x}_0$. Moreover, write the expression of the $\mathcal{Z}$-transform $\mathbf{y}(z)$ of vector $\mathbf{y}(k)$:

$$\mathbf{y}(k) = \qquad\qquad \mathbf{y}(z) =$$

6. Compute, as function of the initial condition $\mathbf{x}_0 = [x_{10},\ x_{20},\ x_{30},\ x_{40}]^{\mathrm{T}}$, the free evolution of the following discrete-time autonomous system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(k), \qquad \mathbf{x}(k) = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix}$$

7. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$:

$\mathbf{F} =$ $\qquad$ $\mathbf{G} =$ $\qquad$ $\mathbf{H} =$

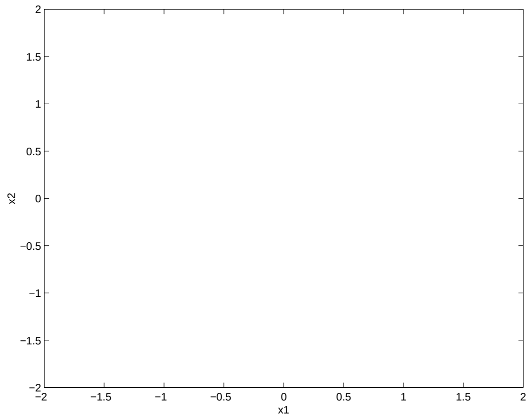
8. Considered a **discrete-time** dynamic system $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k)$ of the second order characterized by two real eigenvalues $\lambda_1 = 0.8$, $\lambda_2 = 1.2$, answer the following questions and draw the qualitative behavior of the state trajectories in the vicinity of the origin:

○ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are real and different.

○ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are straight trajectories of the system.

○ for $t \to \infty$ all the trajectories tend to flatten on one of the two eigenvectors.

○ for $t \to \infty$ all the trajectories tend to zero.



Which name is typically used for denoting the type of trajectories shown above:

○ Node? ○ Focus? ○ Saddle? | ○ Degenerate? ○ Stable? ○ Unstable?

9. Draw the block scheme of the following continuous-time system where $\mathbf{x}_c$ denotes the vector $\mathbf{x}_c = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$.

$$\begin{cases} \dot{\mathbf{x}}_c(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix} \mathbf{x}_c(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \mathbf{x}_c(t) \end{cases}$$

10. Given the dynamic system shown below, write the transfer function $G(z)$ which links the transform $U(z)$ of the input $u(k)$ to the transform $Y(z)$ of the output $y(k)$:

$G(z) =$

$$\begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} 0 & 0 & 0 & -5 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 5 \\ 3 \\ 2 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 7 \end{bmatrix} u(k) \end{cases}$$

11. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ which move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the solution $\mathbf{u}$ which minimizes the Euclidean norm:

$$\mathbf{u} =$$

12. Given a system $(\mathbf{A},\ \mathbf{b})$ completely reachable. The corresponding sampled system (being T the sampling period) is completely reachable if and only if for each couple λ_i, λ_j of different eigenvalues of matrix $\mathbf{A}$ having the same real part it is:

13. Given the following nonlinear differential equations in the state space:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -3\,x_1 \sin^2 x_3 + 2\,x_2^3 - 5x_1x_3^2 + u(t) \end{cases}$$

Set $[x_1\ \ x_2\ \ x_3]^{\mathrm{T}} = [y(t)\ \ \dot{y}(t)\ \ \ddot{y}(t)]^{\mathrm{T}}$, write the corresponding third order nonlinear differential equation which links the input $u(t)$ to the output $y(t)$:

$$\dots$$

14. Write the structure of the matrix $\mathbf{P}^{-1}$ of the state space transformation $\mathbf{x} = \mathbf{P}\overline{\mathbf{x}}$ which brings a not-observable system in the observability standard form:

$$\mathbf{P}^{-1} =$$

Moreover, write the block structure of the obtained matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$:

$$\overline{\mathbf{A}} = \left[\quad \quad \quad \right], \qquad \overline{\mathbf{B}} = \left[\quad \right], \qquad \overline{\mathbf{C}} = \left[\quad \quad \quad \right]$$

Write the simplified form of the transfer matrix $\mathbf{H}(s)$ of the system $\mathcal{S}$ as a function of the submatrices $\mathbf{A}_{i,j}$, $\mathbf{B}_i$ and $\mathbf{C}_j$ which characterize the system $\bar{\mathcal{S}} = (\overline{\mathbf{A}},\ \overline{\mathbf{B}},\ \overline{\mathbf{C}})$:

$$\mathbf{H}(s) =$$

15. a) Write the explicit form of the Ackermann formula for the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of a feedback system:

$$\mathbf{k}^{\mathrm{T}} =$$

b) Write the structures of the desired polynomial $p(\lambda)$ and the matrix $p(\mathbf{A})$ of a continuous-time system when $n = 4$, the desired settling time is $T_a = 6$ s, and all the desired eigenvalues must be located in the same real point λ:

$$\lambda = \qquad\qquad p(\lambda) = \qquad\qquad p(\mathbf{A}) =$$

16. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of a "*reduced order state estimator*":

$$\hat{\mathbf{x}}(t) = \mathbf{T}\begin{bmatrix} \quad \end{bmatrix}$$

$$\dot{\hat{\mathbf{v}}}(t) =$$

17. Given a continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$:

a) it is possible to use an "*open loop state estimator*" if and only if:

...

b) it is possible to use a "*full order closed loop state estimator*" if and only if:

...

18. Write the structure of the dual system $\mathcal{S}_D$ corresponding to a given system $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$:

$$\mathcal{S}_D = (\qquad , \qquad , \qquad , \qquad)$$

19. Write the structure of the transformation matrix $\mathbf{P}_c^{-1}$ which brings an observable system $\mathcal{S} = (\mathbf{A},\ \mathbf{b},\ \mathbf{c})$ in observability canonical form ($\mathbf{x} = \mathbf{P}_c\mathbf{x}_c$):

$$\mathbf{P}_c^{-1} =$$

where α_i are ...

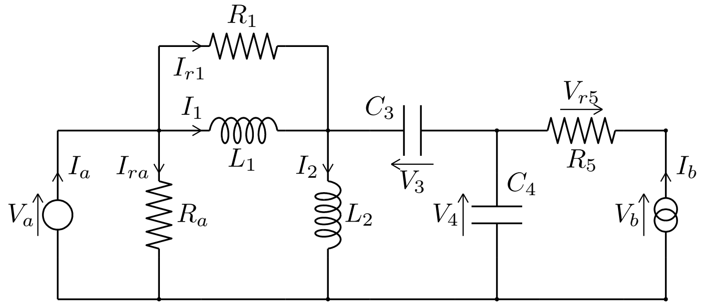
20. Consider a continuous-time nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ and let $\mathbf{x}_0$ be an equilibrium point of the system for constant input $\mathbf{u}_0$. Write the linear part of the Taylor series expansion of the function $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ in the neighborhood of point $(\mathbf{x}_0,\ \mathbf{u}_0)$:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) =$$

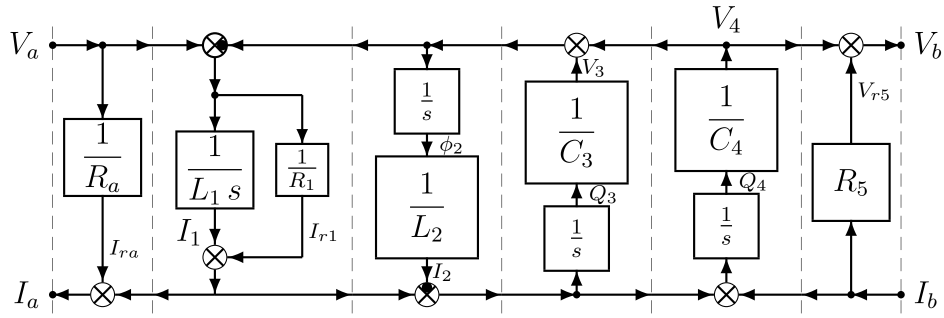
21. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Mechanical Translational*. Moreover, write the constitutive relation (linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols / Names	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$ q_1 v_1				
$\mathcal{D}_2$ q_2 v_2				
$\mathcal{R}$				

22. Consider the following electric circuit composed by the inductances L_1, L_2, the capacities C_3, C_4 and the resistances R_a, R_1 and R_5. Two inputs act on the system: the voltage V_a and the current I_b. The outputs of the system are: the current I_a and the voltage V_b.



The POG model of the given electric scheme has the following structure:



Let $\mathbf{x} = \begin{bmatrix} I_1 & I_2 & V_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \quad & \quad \\ & \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

23. Write the direct Lyapunov stability criterion for **discrete-time** systems.
Consider the nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.
1) If ...

24. Which of the following functions $V(x_1, x_2)$ are positive definite in the neighborhood **of point (1, 1)**:

○ $V(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2;$

○ $V(x_1, x_2) = (x_1^2 - 1)x_2^2 + (x_2^2 - 1)x_1^2;$

○ $V(x_1, x_2) = (x_1^2 - 1) + (x_2^2 - 1);$

○ $V(x_1, x_2) = (x_1 - 1)^2 x_2^2 + (x_2 - 1)^2 x_1^2;$

25. Given the following nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, **continuous-time** and autonomous:

$$\begin{cases} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - \alpha)x_2 \end{cases}$$

a) Compute the Jacobian $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and the matrix $\mathbf{A}_1$ of the linearized system in the neighborhood of the equilibrium point $\overline{\mathbf{x}}_1 = (0, 0)$:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \left[\quad \quad \quad \right] \qquad \mathbf{A}_1 = \left[\quad \quad \right].$$

b) Study, for varying α, the stability of the nonlinear system in the neighborhood of the equilibrium point $\overline{\mathbf{x}}_1$ using the Lyapunov reduced criterion:

26. Given the following nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x})$, **discrete-time** and autonomous:

$$\begin{cases} x_1(k+1) &= -x_2 \\ x_2(k+1) &= x_1 + (x_1^2 - \alpha)x_2 \end{cases}$$

a) Compute, for varying α, the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ of the nonlinear system:

$\overline{\mathbf{x}}_1 = (\quad , \quad),$ $\qquad$ $\overline{\mathbf{x}}_2 = (\quad , \quad),$ $\qquad$ $\overline{\mathbf{x}}_3 = (\quad , \quad)$

b) Study, for varying α, the stability of the nonlinear system in the neighborhood of point $\overline{\mathbf{x}}_1 = (0, 0)$ using the Lyapunov reduced criterion:

c) Given the following Lyapunov function: $V(\mathbf{x}) = x_1^2 + x_2^2$ compute the function $\Delta V(\mathbf{x}(k))$: