System and Control Theory
Test of Genuary 11, 2016
Questions and Exercises

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1. Write the general solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ starting from the initial condition $\mathbf{x}(0)$ at time $h = 0$:

$$\mathbf{x}(k) = \mathbf{A}^k\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{(k-j-1)}\mathbf{B}\mathbf{u}(j)$$

2. Write the discrete time behavior of the output function $\mathbf{y}(t)$, solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and the static equation $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(0)$ at time $t_0 = 0$:

$$\mathbf{y}(t) = \mathbf{C}\, e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

3. A dynamic system characterized by the state function $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k))$

⊗ is a dynamic system; ○ is a linear system;
○ is a continuous-time system; ⊗ is a time-invariant system;

4. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

The system is: ○ reachable? ⊗ not-reachable? ⊗ observable? ○ not-observable?

Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \mathrm{Im}\,[\mathcal{B}_R] = \mathrm{Im}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathcal{E}^- = \mathrm{Im}\,[\mathcal{B}_O] = \mathrm{Im}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

5. Given a discrete-time linear system: $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ and $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$. Write the expression of the output *free evolution* $\mathbf{y}(k)$ of the system starting from the initial condition $\mathbf{x}_0$. Moreover, write the expression of the $\mathcal{Z}$-transform $\mathbf{y}(z)$ of vector $\mathbf{y}(k)$:

$$\mathbf{y}(k) = \mathbf{C}\mathbf{A}^k\mathbf{x}_0, \qquad \mathbf{y}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}z\mathbf{x}_0$$

6. Compute, as function of the initial condition $\mathbf{x}_0 = [x_{10},\ x_{20},\ x_{30},\ x_{40}]^{\mathrm{T}}$, the free evolution of the following discrete-time autonomous system:

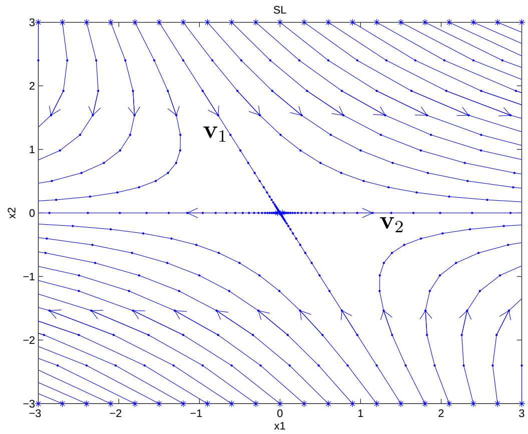
$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{x}(k), \qquad \mathbf{x}(k) = \begin{bmatrix} 1 & k & \frac{k(k-1)}{2} & \frac{k(k-1)(k-2)}{6} \\ 0 & 1 & k & \frac{k(k-1)}{2} \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix}$$

7. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$:

$$\mathbf{F} = e^{\mathbf{A}T}, \qquad \mathbf{G} = \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\, d\sigma, \qquad \mathbf{H} = \mathbf{C}$$

8. Considered a **discrete-time** dynamic system $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k)$ of the second order characterized by two real eigenvalues $\lambda_1 = 0.8$, $\lambda_2 = 1.2$, answer the following questions and draw the qualitative behavior of the state trajectories in the vicinity of the origin:

- ⊗ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are real and different.
- ⊗ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are straight trajectories of the system.
- ⊗ for $t \to \infty$ all the trajectories tend to flatten on one of the two eigenvectors.
- ○ for $t \to \infty$ all the trajectories tend to zero.

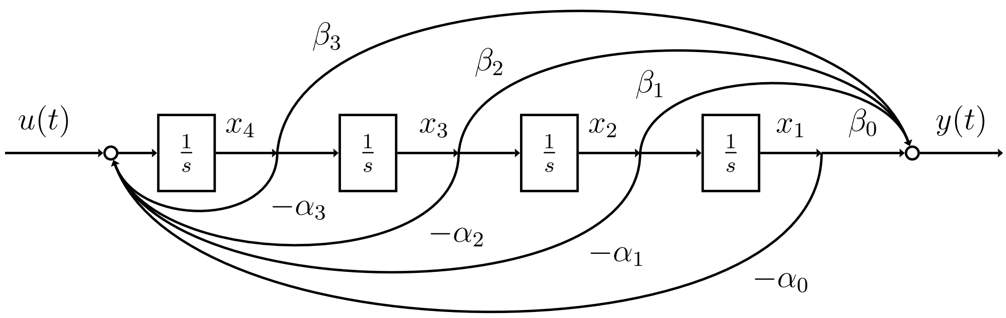


Which name is typically used for denoting the type of trajectories shown above:

○ Node? ○ Focus? ⊗ Saddle? | ○ Degenerate? ○ Stable? ⊗ Unstable?

9. Draw the block scheme of the following continuous-time system where $\mathbf{x}_c$ denotes the vector $\mathbf{x}_c = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$.

$$\begin{cases} \dot{\mathbf{x}}_c(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix} \mathbf{x}_c(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \mathbf{x}_c(t) \end{cases}$$



10. Given the dynamic system shown below, write the transfer function $G(z)$ which links the transform $U(z)$ of the input $u(k)$ to the transform $Y(z)$ of the output $y(k)$:

$$G(z) = \frac{2z^3 + 3z^2 + 5z + 1}{z^4 + 4z^3 + 3z^2 + z + 5} + 7$$

$$\begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} 0 & 0 & 0 & -5 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 5 \\ 3 \\ 2 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 7 \end{bmatrix} u(k) \end{cases}$$

11. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ which move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the solution $\mathbf{u}$ which minimizes the Euclidean norm:

$$\mathbf{u} = (\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}[\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0)]$$

12. Given a system $(\mathbf{A},\ \mathbf{b})$ completely reachable. The corresponding sampled system (being T the sampling period) is completely reachable if and only if for each couple λ_i, λ_j of different eigenvalues of matrix $\mathbf{A}$ having the same real part it is:

$$\mathrm{Im}(\lambda_i - \lambda_j) \neq \frac{2k\pi}{T} \qquad\qquad k = \pm 1,\ \pm 2,\ \ldots$$

13. Given the following nonlinear differential equations in the state space:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -3\,x_1 \sin^2 x_3 + 2\,x_2^3 - 5x_1x_3^2 + u(t) \end{cases}$$

Set $[x_1\ \ x_2\ \ x_3]^{\mathrm{T}} = [y(t)\ \ \dot{y}(t)\ \ \ddot{y}(t)]^{\mathrm{T}}$, write the corresponding third order nonlinear differential equation which links the input $u(t)$ to the output $y(t)$:

$$\dddot{y}(t) + 3\,y(t)\sin^2 \ddot{y}(t) - 2\,\dot{y}^3(t) + 5y(t)\ddot{y}^2(t) = u(t).$$

14. Write the structure of the matrix $\mathbf{P}^{-1}$ of the state space transformation $\mathbf{x} = \mathbf{P}\overline{\mathbf{x}}$ which brings a not-observable system in the observability standard form:

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} \qquad \text{where} \qquad \mathrm{Im}\mathbf{P}_1^{\mathrm{T}} = \mathrm{Im}(\mathcal{O}^-)^{\mathrm{T}} \text{ and } \mathbf{P}_2 \text{ makes non singular the matrix } \mathbf{P}^{-1}.$$

Moreover, write the block structure of the obtained matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$:

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{1,1} & 0 \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix} \qquad \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & 0 \end{bmatrix}$$

Write the simplified form of the transfer matrix $\mathbf{H}(s)$ of the system $\mathcal{S}$ as a function of the submatrices $\mathbf{A}_{i,j}$, $\mathbf{B}_i$ and $\mathbf{C}_j$ which characterize the system $\bar{\mathcal{S}} = (\overline{\mathbf{A}},\ \overline{\mathbf{B}},\ \overline{\mathbf{C}})$:

$$\mathbf{H}(s) = \mathbf{C}_1(s\,\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{B}_1$$

15. a) Write the explicit form of the Ackermann formula for the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of a feedback system:

$$\mathbf{k}^{\mathrm{T}} = -\begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} (\mathcal{R}^+)^{-1} p(\mathbf{A})$$

b) Write the structures of the desired polynomial $p(\lambda)$ and the matrix $p(\mathbf{A})$ of a continuous-time system when $n = 4$, the desired settling time is $T_a = 6$ s, and all the desired eigenvalues must be located in the same real point λ:

$$\lambda = -\frac{3}{T_a} = -0.5, \qquad p(\lambda) = (\lambda + 0.5)^4, \qquad p(\mathbf{A}) = (\mathbf{A} + 0.5\mathbf{I})^4$$

16. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of a "*reduced order state estimator*":

$$\hat{\mathbf{x}}(t) = \mathbf{T}\begin{bmatrix} \hat{\mathbf{v}}(t) - \mathbf{L}\mathbf{y}(t) \\ \mathbf{y}(t) \end{bmatrix}$$

$$\dot{\hat{\mathbf{v}}}(t) = (\overline{\mathbf{A}}_{11} + \mathbf{L}\overline{\mathbf{A}}_{21})\hat{\mathbf{v}}(t) + (\overline{\mathbf{A}}_{12} + \mathbf{L}\overline{\mathbf{A}}_{22} - \overline{\mathbf{A}}_{11}\mathbf{L} - \mathbf{L}\overline{\mathbf{A}}_{21}\mathbf{L})\mathbf{y}(t) + (\mathbf{B}_1 + \mathbf{L}\mathbf{B}_2)\mathbf{u}(t)$$

17. Given a continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$:

    a) it is possible to use an "*open loop state estimator*" if and only if:
    the system is asymptotically stable

    b) it is possible to use a "*full order closed loop state estimator*" if and only if:
    the not-observable part of the system is asymptotically stable

18. Write the structure of the dual system $\mathcal{S}_D$ corresponding to a given system $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$:

$$\mathcal{S}_D = (\mathbf{A}^{\mathrm{T}},\ \mathbf{C}^{\mathrm{T}},\ \mathbf{B}^{\mathrm{T}},\ \mathbf{D}^{\mathrm{T}})$$

19. Write the structure of the transformation matrix $\mathbf{P}_c^{-1}$ which brings an observable system $\mathcal{S} = (\mathbf{A},\ \mathbf{b},\ \mathbf{c})$ in observability canonical form ($\mathbf{x} = \mathbf{P}_c\mathbf{x}_c$):

$$\mathbf{P}_c^{-1} = (\mathcal{O}_c^-)^{-1}\mathcal{O}^- = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \dots & \dots & 1 & 0 \\ \alpha_3 & \dots & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \dots & \dots & 0 & 0 \\ 1 & 0 & \dots & \dots & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathbf{A} \\ \mathbf{c}\mathbf{A}^2 \\ \dots \\ \mathbf{c}\mathbf{A}^{n-1} \end{bmatrix}$$

    where α_i are the coefficients of the characteristic polynomial of matrix $\mathbf{A}$.

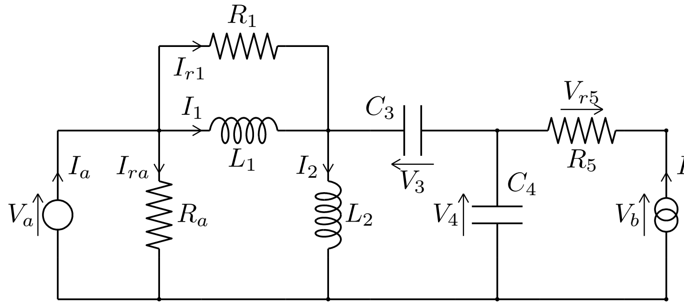
20. Consider a continuous-time nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ and let $\mathbf{x}_0$ be an equilibrium point of the system for constant input $\mathbf{u}_0$. Write the linear part of the Taylor series expansion of the function $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ in the neighborhood of point $(\mathbf{x}_0,\ \mathbf{u}_0)$:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \underbrace{\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)}_{0} + \left.\frac{\partial \mathbf{f}(\mathbf{x},\ \mathbf{u})}{\partial\,\mathbf{x}}\right|_{(\mathbf{x}_0,\mathbf{u}_0)}(\mathbf{x}-\mathbf{x}_0) + \left.\frac{\partial \mathbf{f}(\mathbf{x},\ \mathbf{u})}{\partial\,\mathbf{u}}\right|_{(\mathbf{x}_0,\mathbf{u}_0)}(\mathbf{u}-\mathbf{u}_0) + \mathbf{h}_1(\mathbf{x}, \mathbf{u})$$

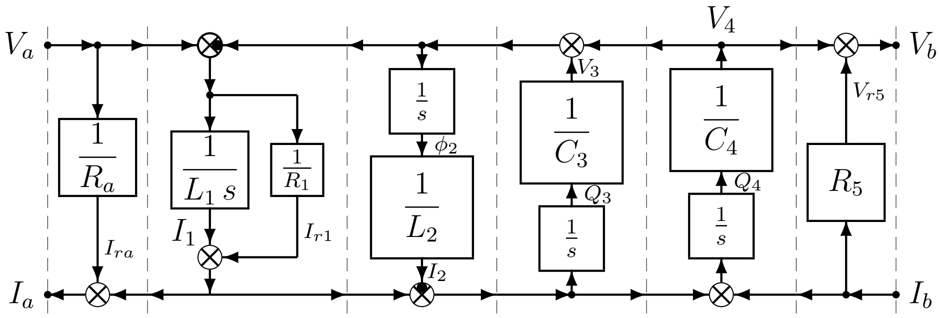
21. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Mechanical Translational*. Moreover, write the constitutive relation (linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$ q_1 v_1	M Mass P Momentum $\dot{x}$ Velocity	$P = \Phi_M(\dot{x})$	$P = M\,\dot{x}$	$\frac{d\,P}{dt} = F$
$\mathcal{D}_2$ q_2 v_2	E Elasticity x Displacement F Force	$x = \Phi_E(F)$	$x = E\,F$	$\frac{d\,x}{dt} = \dot{x}$
$\mathcal{R}$	b Friction	$F = \Phi_b(\dot{x})$	$F = b\,\dot{x}$	

22. Consider the following electric circuit composed by the inductances L_1, L_2, the capacities C_3, C_4 and the resistances R_a, R_1 and R_5. Two inputs act on the system: the voltage V_a and the current I_b. The outputs of the system are: the current I_a and the voltage V_b.



The POG model of the given electric scheme has the following structure:



Let $\mathbf{x} = \begin{bmatrix} I_1 & I_2 & V_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & -\frac{1}{R_1} & -\frac{1}{R_1} \\ 1 & -1 & -\frac{1}{R_1} & -\frac{1}{R_1} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \frac{1}{R_1} & 0 \\ \frac{1}{R_1} & 1 \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & -\frac{1}{R_1} & -\frac{1}{R_1} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_a} + \frac{1}{R_1} & 0 \\ 0 & R_5 \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

23. Write the direct Lyapunov stability criterion for **discrete-time** systems.

Consider the nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.

1) If in a neighborhood W *of point* $\mathbf{x}_0$ *exists a* positive definite *continuous function* $V(\mathbf{x}) : W \to \mathcal{R}$, *and if the function* $\Delta V(\mathbf{x})$ *is* negative semidefinite, *then the point* $\mathbf{x}_0$ *is* stable. *If the function* $\Delta V(\mathbf{x})$ *is* negative definite, *then the point* $\mathbf{x}_0$ *is* asymptotically stable.

24. Which of the following functions $V(x_1, x_2)$ are positive definite in the neighborhood **of point (1, 1)**:

- ⊗ $V(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$;
- ○ $V(x_1, x_2) = (x_1^2 - 1)x_2^2 + (x_2^2 - 1)x_1^2$;
- ○ $V(x_1, x_2) = (x_1^2 - 1) + (x_2^2 - 1)$;
- ⊗ $V(x_1, x_2) = (x_1 - 1)^2 x_2^2 + (x_2 - 1)^2 x_1^2$;

25. Given the following nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, **continuous-time** and autonomous:

$$\begin{cases} \dot{x}_1 &=& -x_2 \\ \dot{x}_2 &=& x_1 + (x_1^2 - \alpha)x_2 \end{cases}$$

a) Compute the Jacobian $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and the matrix $\mathbf{A}_1$ of the linearized system in the neighborhood of the equilibrium point $\overline{\mathbf{x}}_1 = (0, 0)$:

The Jacobian $\mathbf{A}(\mathbf{x})$ and the matrix $\mathbf{A}_1$ of the linearized system in the neighborhood of $\overline{\mathbf{x}}_1$ are the following:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} 0 & -1 \\ 1 + 2x_1x_2 & x_1^2 - \alpha \end{bmatrix}, \qquad \mathbf{A}_1 = \begin{bmatrix} 0 & -1 \\ 1 & -\alpha \end{bmatrix}.$$

b) Study, for varying α, the stability of the nonlinear system in the neighborhood of the equilibrium point $\overline{\mathbf{x}}_1$ using the Lyapunov reduced criterion:

The characteristic polynomial of matrix $\mathbf{A}_1$ is the following:

$$\Delta_{\mathbf{A}_1}(s) = s^2 + \alpha s + 1 = 0$$

Using the Lyapunov reduced criterion it can be stated that the equilibrium point $\overline{\mathbf{x}}_1 = (0, 0)$ of the nonlinear system: a) is asymptotically stable if $\alpha > 0$; b) is unstable if $\alpha < 0$; c) the criterion cannot be used for $\alpha = 0$.

26. Given the following nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x})$, **discrete-time** and autonomous:

$$\begin{cases} x_1(k+1) &=& -x_2 \\ x_2(k+1) &=& x_1 + (x_1^2 - \alpha)x_2 \end{cases}$$

a) Compute, for varying α, the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ of the nonlinear system:

The equilibrium points can be determined imposing $x_1(k+1) = x_1(k)$ and $x_2(k+1) = x_2(k)$:

$$x_1 = -x_2, \qquad \Rightarrow \qquad x_2 = -x_2 + (x_2^2 - \alpha)x_2 \qquad \Leftrightarrow \qquad (x_2^2 - \alpha - 2)x_2 = 0$$

From the last relation it is $x_2 = 0$, $x_2 = \sqrt{\alpha + 2}$ and $x_2 = -\sqrt{\alpha + 2}$ from which one obtains:

$$\overline{\mathbf{x}}_1 = (0,\ 0), \qquad \overline{\mathbf{x}}_2 = (\sqrt{\alpha+2},\ -\sqrt{\alpha+2}), \qquad \overline{\mathbf{x}}_3 = (-\sqrt{\alpha+2},\ \sqrt{\alpha+2}).$$

b) Study, for varying α, the stability of the nonlinear system in the neighborhood of point $\overline{\mathbf{x}}_1 = (0, 0)$ using the Lyapunov reduced criterion:

The Jacobian $\mathbf{A}(\mathbf{x})$, the matrix $\mathbf{A}_1$ and the characteristic polynomial of matrix $\mathbf{A}_1$ are equal to those computed in the previous point:

$$\Delta_{\mathbf{A}_1}(z) = z^2 + \alpha z + 1 = 0 \qquad \rightarrow z_{1,2} = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - 1}$$

Using the Lyapunov reduced criterion it can be stated that the equilibrium point $\overline{\mathbf{x}}_1$: a) is unstable if $|\alpha| > 2$; b) the criterion cannot be used for $|\alpha| < 2$. In fact, for $|\alpha| < 1$ the two solutions $z_{1,2}$ of the characteristic polynomial are located on the unitary circle, that is they have unitary module.

c) Given the following Lyapunov function: $V(\mathbf{x}) = x_1^2 + x_2^2$ compute the function $\Delta V(\mathbf{x}(k))$:

The function $\Delta V(\mathbf{x}(k))$ has the following form:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= (-x_2)^2 + (x_1 + (x_1^2 - \alpha)x_2)^2 - x_1^2 - x_2^2 \\ &= x_2^2 + x_1^2 + (x_1^2 - \alpha)^2 x_2^2 + 2x_1x_2(x_1^2 - \alpha) - x_1^2 - x_2^2 \\ &= x_2^2(x_1^2 - \alpha)^2 + 2x_1x_2(x_1^2 - \alpha) = [x_2(x_1^2 - \alpha) + 2x_1]x_2(x_1^2 - \alpha) \end{aligned}$$