System and Control Theory
Test of Dicember 23, 2015
Questions and Exercises

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1. Write the general solution of the following time-varying difference equation $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$, being $\mathbf{x}(h)$ the state at time h:

   $$\mathbf{x}(k) =$$

   where $\mathbf{\Phi}(k,h)$ denotes the state transition matrix of the system.

2. Write the explicit form of the *transition matrix of the system* $\mathbf{\Phi}(t,t_0)$ for continuous-time invariant linear system:

   $$\mathbf{\Phi}(t,t_0) =$$

3. Write the general solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(0)$ at time $t_0 = 0$:

   $$\mathbf{x}(t) =$$

4. Write the symbol used for describing "*the set of the reachable states at time* t_1*, starting from* $\{t_0, \mathbf{x}(t_0)\}$":

   $$\dots$$

5. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

   $$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \quad \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \quad \end{bmatrix}$$

   The system is: ○ reachable? ○ not-reachable? ○ observable? ○ not-observable?

6. The following symbolic representation:

   $$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \end{cases} \qquad \mathbf{x}(t) \in \mathbf{R}^n$$

   is used for describing a system with the following characteristics:

   ○ a dynamic system; ○ a continuous-time system;
   ○ a linear system; ○ a lumped system;
   ○ a time-varying system; ○ a system without inputs;

7. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^{\mathrm{T}}$, the free evolution of the following continuous-time autonomous system:

   $$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t) \qquad \mathbf{x}(t) = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

8. Let $\mathbf{A}$ and $\bar{\mathbf{A}}$ be two similar matrices $\mathbf{A} = \mathbf{T}\bar{\mathbf{A}}\mathbf{T}^{-1}$. The matrix function $\mathbf{A}^k$ satisfies the following property:

○ $\mathbf{A}^k = \mathbf{T}^{-1}\,\bar{\mathbf{A}}^k\,\mathbf{T}$; ○ $\mathbf{A}^k = \mathbf{T}^{-k}\,\bar{\mathbf{A}}^k\,\mathbf{T}^k$; ○ $\mathbf{A}^k = \mathbf{T}\,\bar{\mathbf{A}}^k\,\mathbf{T}^{-1}$; ○ $\mathbf{A}^k = \mathbf{T}^k\,\bar{\mathbf{A}}^k\,\mathbf{T}^{-k}$;

9. Write the formula for computing the state transition matrix $e^{\mathbf{A}t}$ of a continuous-time linear system using the Laplace transform:

$$e^{\mathbf{A}t} =$$

10. Consider the following linear dynamic system:

$$\dot{\mathbf{x}} = \begin{bmatrix} -6 & -1 \\ -4 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\mathbf{u} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.5 & 1 \\ -1 & 2.5 \end{bmatrix}\mathbf{x} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix}\mathbf{x} = \mathbf{C}\,\mathbf{x}.$$

a) Write the formula used for computing the equilibrium point $\mathbf{x}_0$ corresponding to the constant input $\mathbf{u} = \mathbf{u}_0$:

$$\mathbf{x}_0 = \qquad\qquad\qquad\qquad .$$

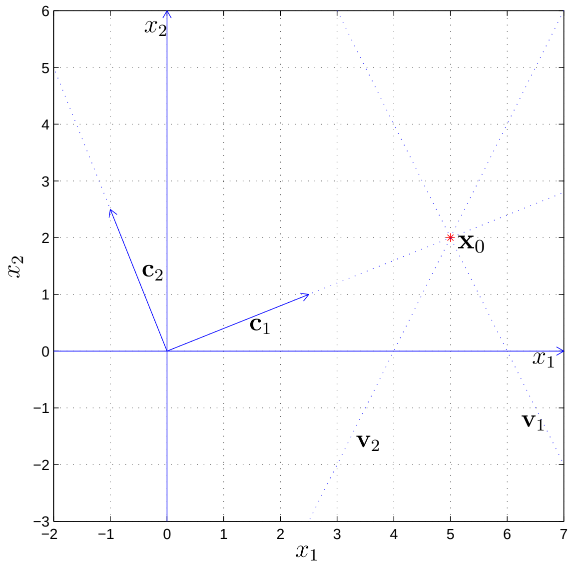
It is easy to verify that for constant input $\mathbf{u}_0 = 16$ the equilibrium point of the given system is $\mathbf{x}_0 = [5,\ 2]^{\mathrm{T}}$. The eigenvalues of matrix A are:

$$\lambda_1 = -4, \qquad \lambda_2 = -8$$

The corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

b) In the box on the right draw the qualitative trajectory of the dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}$ starting from the initial condition $\mathbf{x}(0) = 0$ and when the constant input is $\mathbf{u}_0 = 16$.



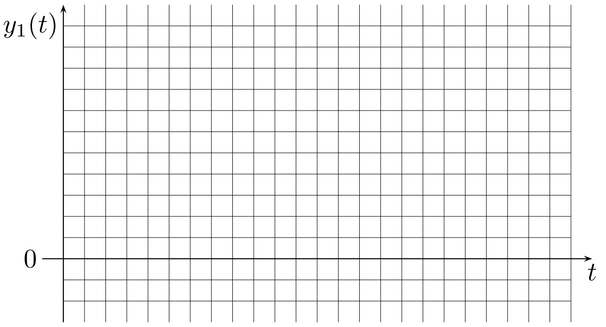
c) Which type of trajectory characterizes the time response of the system when $\mathbf{u}_0 = 16$:

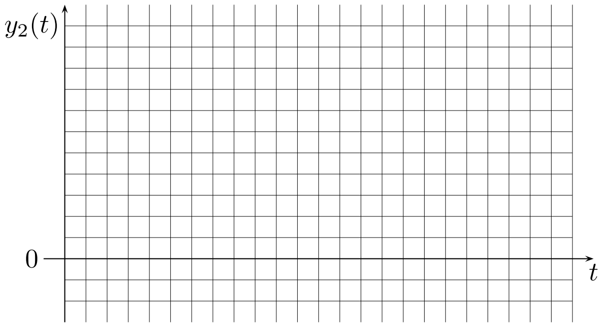
○ Node? ○ Degen. Node? ○ Focus? ○ Saddle? ○ Stable? ○ Unstable?

d) Compute the settling time T_a and the period T_ω of the oscillation of the time response $\mathbf{x}(t)$:

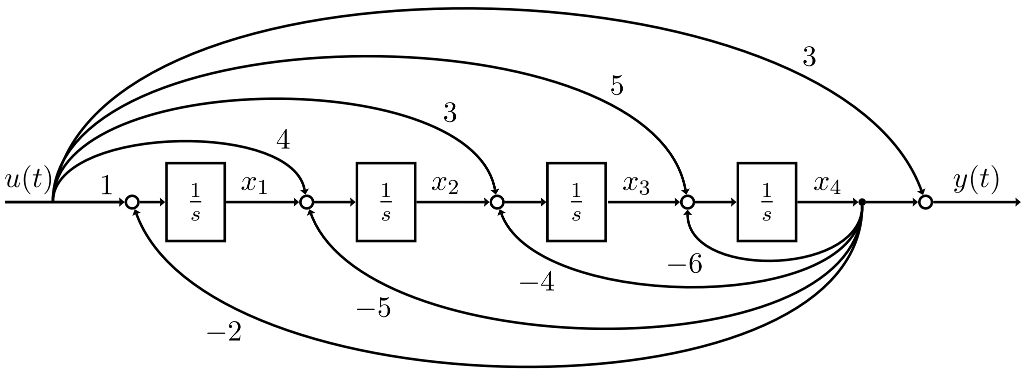
$$T_a = \qquad\qquad\qquad\qquad T_\omega =$$

e) Draw the qualitative behavior of the time response of the two outputs $y_1(t)$ and $y_2(t)$:


$y_1(t)$
0
t


$y_2(t)$
0
t

11. Given following block scheme:



Compute the transfer function of the system:

$$G(s) = \frac{Y(s)}{U(s)} = \dots$$

Without further calculations it is possible to state that the given system:

- ◯ is surely stable;
- ◯ is completely observable;
- ◯ is completely reachable;
- ◯ is in the observability canonical form;
- ◯ is in the reachability standard form;
- ◯ is a system constructable;

12. Given a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$ completely reachable and with only one input. Let $\Delta_{\mathbf{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$ be the characteristic polynomial of matrix $\mathbf{A}$ and let $p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0$ be a monic polynomial freely chosen. Write the expression of vector $\mathbf{k}^{\mathrm{T}}$ which using the static feedback $\mathbf{u} = \mathbf{k}^{\mathrm{T}}\mathbf{x}$ is able to match the eigenvalues of matrix $\mathbf{A} + \mathbf{b}\mathbf{k}^{\mathrm{T}}$ with the roots of polynomial $p(\lambda)$:

$$\mathbf{k}^{\mathrm{T}} =$$

where $\mathbf{k}_c^{\mathrm{T}} = [\qquad\qquad\qquad\qquad]$.

13. Given the following discrete-time linear system:

$$\begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 0 & 0 & 0.5 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(k) \end{cases}$$

Thinking to the block structure of the systems in standard form it is possible to state that:

- ◯ the system is in the standard observability form;
- ◯ the system is in the reachability standard form;
- ◯ for this system it is possibile to build an asymptotic state observer;
- ◯ the system can be stabilized using a static state feedback;

14. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ which move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the solution $\mathbf{u}$ which minimizes the Euclidean norm:

$$\mathbf{u} =$$

15. Write the "*separation property*" of the regulator:

16. The Ackermann formula for computing the gain vector $\mathbf{l}$ of an asymptotic state observer which freely places the eigenvalues of matrix $\mathbf{A} + \mathbf{lc}$ can be used:

- ○ only if the system is reachable;
- ○ only if the system is observable;
- ○ only for sistemi ad una sola uscita;
- ○ only for systems with one input;
- ○ only if polynomial $\det(s\mathbf{I} - \mathbf{A})$ is known;
- ○ only if polynomial $p(\lambda)$ is known;

17. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of:
a) an **open loop** state estimator and the time evolution of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ obtained starting from the initial condition $\mathbf{e}(0)$:

$\dot{\hat{\mathbf{x}}}(t) =$ $\qquad\qquad$ $\mathbf{e}(t) =$

b) uno stimatore asintotico of the system **in catena chiusa of order pieno** and the time evolution of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ obtained starting from the initial condition $\mathbf{e}(0)$:

$\dot{\hat{\mathbf{x}}}(t) =$ $\qquad\qquad$ $\mathbf{e}(t) =$

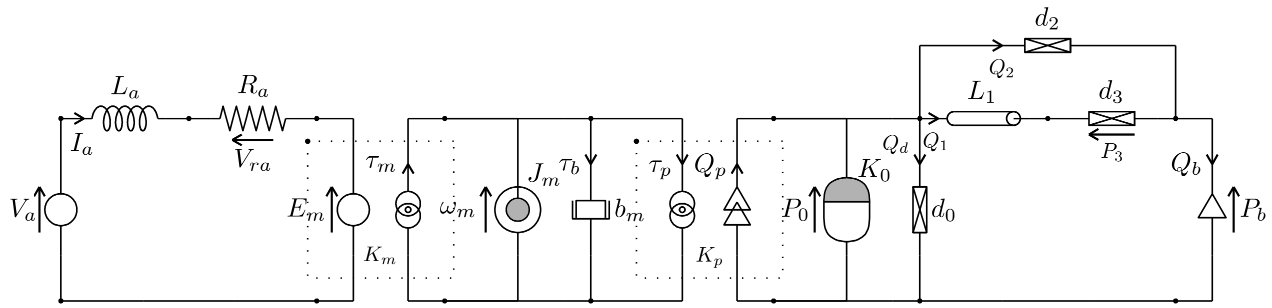
18. Write, within the following table, the symbols and the names of the energy variables and power variables that characterize the Energetic Domain: *Mechanical Rotational*. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols / Names	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$ q_1 v_1				
$\mathcal{D}_2$ q_2 v_2				
$\mathcal{R}$				

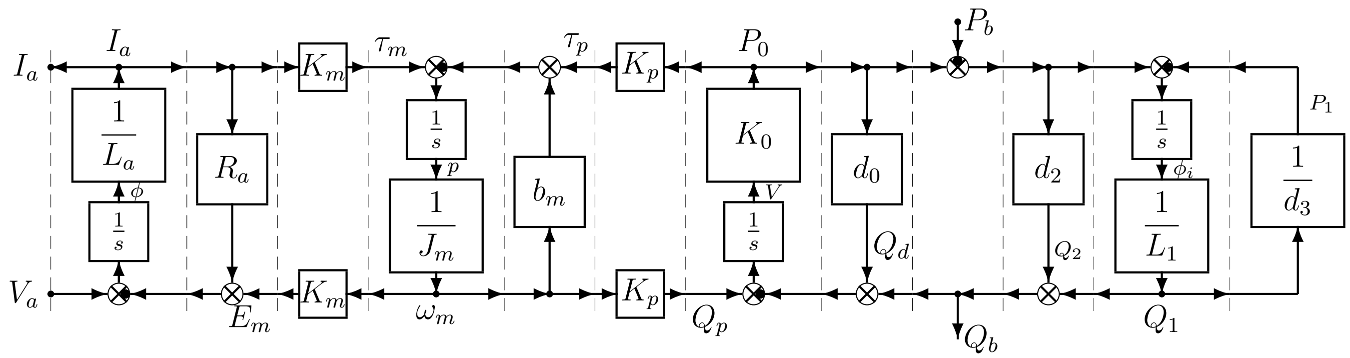
19. Which of the following functions $V(x_1, x_2)$ are positive definite in the neighborhood **of point (1, 0)**:

- ○ $V(x_1, x_2) = x_1^2 + (x_2 - 1)^2$;
- ○ $V(x_1, x_2) = (x_1^2 - 1)(x_2^2 + 1) + x_2^2$;
- ○ $V(x_1, x_2) = (x_1 - 1)^2 + x_2^2$;
- ○ $V(x_1, x_2) = (x_1^2 + 1)(x_2^2 - 1) + x_1^2$;

20. Consider the following dynamic system composed by a DC electric motor which moves an hydraulic pump which in turn feeds an hydraulic load: L_a, R_a, K_m, J_m and b_m are the parameters of the DC electric motor; K_p, K_0, L_1, d_0, d_2 and d_3 are the parameters of the gear pump and the hydraulic load. Two inputs act on the system: the voltage V_a and the pressure P_b. The outputs of the system are: the current I_a and the volume flow rate Q_b.



The POG model of the given dynamic system has the following structure:



Let $\mathbf{x} = \begin{bmatrix} I_a & \omega_m & P_0 & Q_1 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} V_a & P_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} I_a & Q_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_a \\ \dot{\omega}_m \\ \dot{P}_0 \\ \dot{Q}_1 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_a \\ \omega_m \\ P_0 \\ Q_1 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ P_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ Q_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ P_b \end{bmatrix}}_{\mathbf{u}}$$

21. Write the La Salle - Krasowskii stability criterion for continuous-time systems.

Consider the nonlinear continuous-time system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.

If . . .

22. Given the following nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, continuous-time and autonomous:

$$\begin{cases} \dot{x}_1 &= \alpha x_1 - x_1 x_2 \\ \dot{x}_2 &= x_1^2 - \beta x_2 \end{cases}$$

a) Compute, as a function of α and β, the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ of the system:

$\overline{\mathbf{x}}_1 = (\quad , \quad)$, $\qquad \overline{\mathbf{x}}_2 = (\quad , \quad)$, $\qquad \overline{\mathbf{x}}_3 = (\quad , \quad)$

b) Compute the Jacobian $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ of the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

c) Compute the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ of the linearized system in the neighborhood of the equilibrium points:

$$\mathbf{A}_1 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \qquad \mathbf{A}_3 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

d) Study, for varying α and β, the stability of the nonlinear system in the neighborhood of the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ using the reduced Lyapunov criterion:

e) For $\alpha = 0$ and $\beta > 0$ study the stability of the nonlinear system in the neighborhood of point $\overline{\mathbf{x}} = (0, 0)$ using the "direct" Lyapunov criterion and the following Lyapunov function: $V(\mathbf{x}) = x_1^2 + x_2^2$. Eventually, use the La Salle - Krasowskii criterion.