System and Control Theory
Test of February 5, 2015
Questions and Exercises

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1. Write the closed form solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$:

$$\mathbf{x}(t) =$$

2. Write the discrete time behavior of the output function $\mathbf{y}(k)$, solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ and the static equation $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$ starting from the initial condition $\mathbf{x}(0)$ at time $h = 0$:

$$\mathbf{y}(k) =$$

3. Describe what the symbol $\mathcal{X}^+(k)$ represents for discrete systems $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$:

   ...

   Moreover, write the usual way of computing the set $\mathcal{X}^+(k)$:

$$\mathcal{X}^+(k) =$$

4. Give the meaning of the symbol $\mathcal{E}^+(t_0, t_1, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$:

   ...

5. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} & & \\ & & \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} & & \\ & & \end{bmatrix}$$

   The system is: ○ reachable? ○ not-reachable? ○ observable? ○ not-observable?

   Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \operatorname{Im}[\mathcal{B}_R] = \operatorname{Im}\begin{bmatrix} & \\ & \end{bmatrix}, \qquad \mathcal{E}^- = \operatorname{Im}[\mathcal{B}_O] = \operatorname{Im}\begin{bmatrix} \\ \end{bmatrix}.$$

6. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$:

$$\mathbf{F} = \qquad \mathbf{G} = \qquad \mathbf{H} =$$

7. Given a continuous-time linear system: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$. Apply the Laplace transform to the system and provide the expression of the transform $\mathbf{y}(s)$ of the output vector $\mathbf{y}(t)$ corresponding to the *forced evolution* of the system:

$$\mathbf{y}(s) =$$

8. Applying to the dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ the state space transformation $\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}}$ one obtains a transformed system $\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t)$, $\mathbf{y}(t) = \tilde{\mathbf{C}}\,\tilde{\mathbf{x}}(t)$ characterized by the following matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$:

$\tilde{\mathbf{A}} =$ $\qquad\qquad$ $\tilde{\mathbf{B}} =$ $\qquad\qquad$ $\tilde{\mathbf{C}} =$

9. Consider the following linear dynamic system:

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \mathbf{u} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} \mathbf{x} = \mathbf{C}\,\mathbf{x}.$$

a) Write the formula used for computing the equilibrium point $\mathbf{x}_0$ corresponding to the constant input $\mathbf{u} = \mathbf{u}_0$:

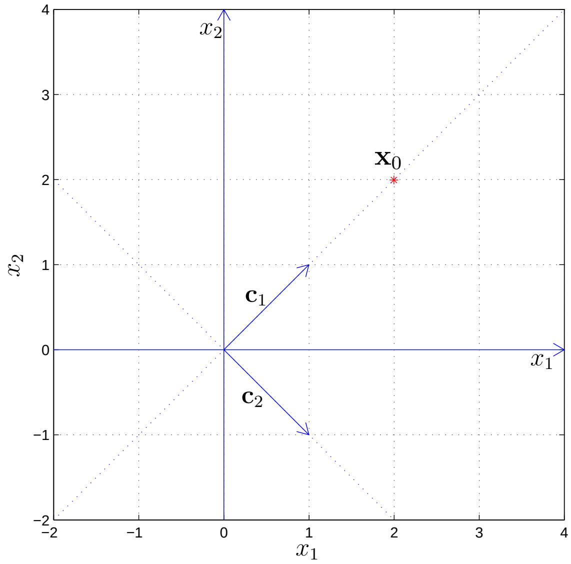
$$\mathbf{x}_0 = \qquad\qquad\qquad\qquad .$$

It is easy to verify that for constant input $\mathbf{u}_0 = 4$ the equilibrium point of the given system is $\mathbf{x}_0 = [2,\ 2]^{\mathrm{T}}$.

b) In the box on the right draw the qualitative trajectory of the dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}$ obtained starting from the zero initial condition $\mathbf{x}(0) = 0$ when the input is constant: $\mathbf{u}_0 = 4$.

c) Compute the eigenvalues of the system:

$\lambda_1 =$ $\qquad\qquad$ $\lambda_2 =$



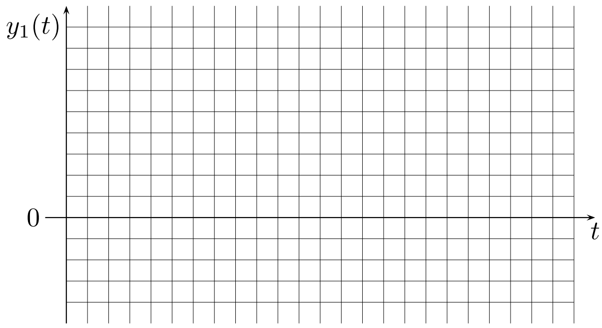
d) Which type of trajectory characterizes the time response of the system when $\mathbf{u}_0 = 4$:

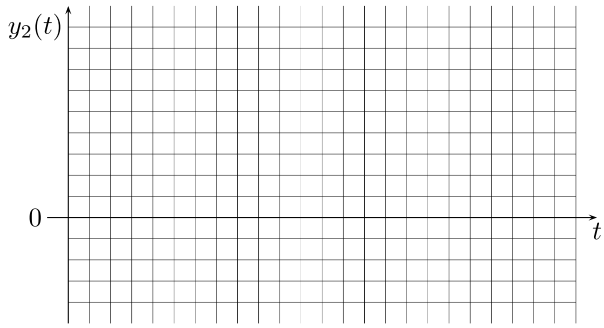
◯ Node? ◯ Degen. Node? ◯ Focus? ◯ Saddle? ◯ Stable? ◯ Unstable?

e) Compute the settling time T_a and the period T_ω of the oscillation of the time response $\mathbf{x}(t)$:

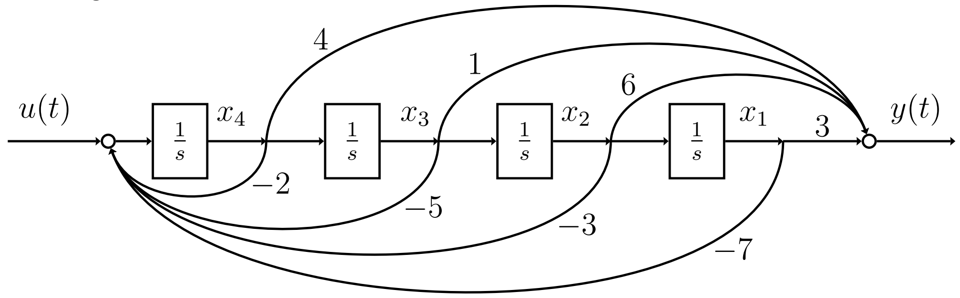
$T_a =$ $\qquad\qquad$ $T_\omega =$

f) Draw the qualitative behavior of the time response of the two outputs $y_1(t)$ and $y_2(t)$:





10. Given following block scheme:



Compute the transfer function of the system:

$$G(s) = \frac{Y(s)}{U(s)} = \ldots$$

Moreover, it is also possible to state that the system $G(s)$:

- ◯ is completely reachable;
- ◯ is completely observable;
- ◯ is surely stable;
- ◯ is in the reachability standard form;
- ◯ is in the observability canonical form;
- ◯ is stabilizable with the feedback $u = \mathbf{k}\,\mathbf{x}$;

11. Given a SISO linear system of the forth order ($n = 4$), completely observable, characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$.
a) Write the structure of the matrices $\mathbf{A}_o$, $\mathbf{b}_o$ and $\mathbf{c}_o$ of the corresponding observability canonical form. Let $p(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ be the characteristic polynomial of matrix $\mathbf{A}$.

$$\mathbf{A}_o = \left[\quad \right], \quad \mathbf{b}_o = \left[\quad \right], \quad \mathbf{c}_o = \left[\quad \right]$$

b) Moreover, write the structure of matrix $\mathbf{P}$ which, together with the space transformation $\mathbf{x} = \mathbf{P}\mathbf{x}_o$, brings the system in the observability canonical form.

$$\mathbf{P} =$$

12. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^{\mathrm{T}}$, the free evolution of the following continuous-time autonomous system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t), \qquad \mathbf{x}(t) = \left[\quad \right] \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

13. For the discret-time linear systems $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,\mathbf{u}(k)$, write the condition that must be satisfied such that it is possible to move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$:

$$\ldots$$

14. Given the following continuous-time linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

Thinking to the block structure of the systems in standard form it is possible to state that:

- ◯ the system is in the standard observability form;
- ◯ the system is in the reachability standard form;
- ◯ for this system it is possibile to build an asymptotic state observer;
- ◯ the system can be stabilized using a static state feedback;

Using the structural properties of the given system compute the transfer function $G(s) = \frac{Y(s)}{U(s)}$ which links the input $U(s) = \mathcal{L}[u(t)]$ to the output $Y(s) = \mathcal{L}[y(t)]$

$G(s) =$

15. Given a system $(\mathbf{A},\ \mathbf{c})$ completely observable. The corresponding sampled system (being T the sampling period) is completely observable if and only if for each couple λ_i, λ_j of eigenvalues of $\mathbf{A}$ having the same real part, it is:

...

16. Given the discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, write the structure of:

a) an *open loop* state estimator:

$\hat{\mathbf{x}}(k+1) =$

b) a *full order closed loop* state estimator:

$\hat{\mathbf{x}}(k+1) =$

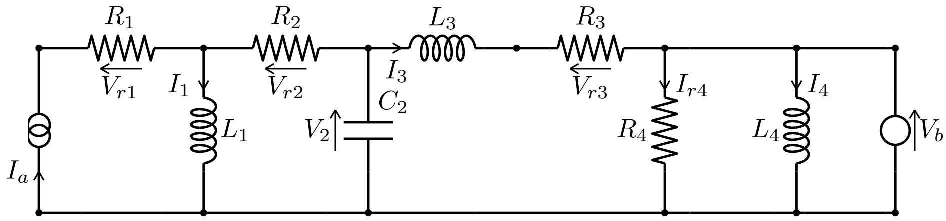
c) the time evolution of the estimation errors $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ in the two previous cases a) and b) starting from the initial condition $\mathbf{e}(0)$:

$\mathbf{e}(k) =$ $\qquad\qquad$ $\mathbf{e}(k) =$

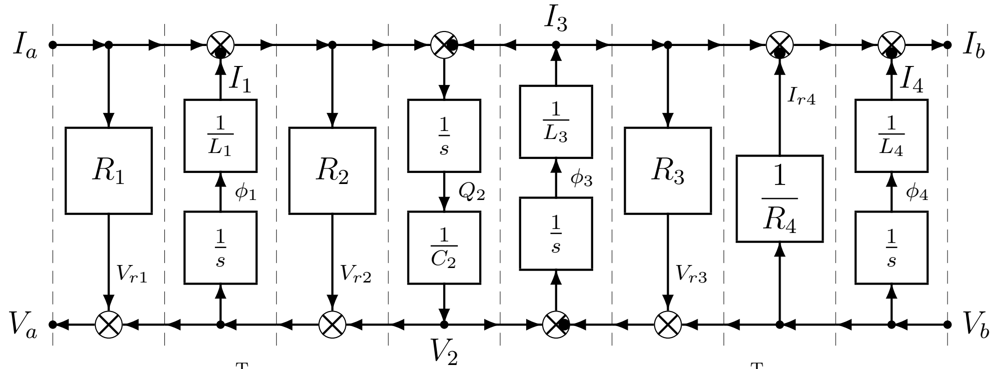
17. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: **Electromagnetic**. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols / Names	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$ q_1 v_1				
$\mathcal{D}_2$ q_2 v_2				
$\mathcal{R}$				

18. Consider the following electric circuit composed by the inductances L_1, L_3, L_4, the capacity C_2 and the resistances R_1, R_2, R_3 and R_4. Two inputs act on the system: the current I_a and the voltage V_b. The outputs of the system are: the voltage V_a and the current I_b.



The POG model of the given electric circuit has the following structure:



Let $\mathbf{x} = [\, I_1 \;\; V_2 \;\; I_3 \;\; I_4 \,]^{\mathrm{T}}$ be the state vector, $\mathbf{u} = [\, I_a \;\; V_b \,]^{\mathrm{T}}$ the input vector and $\mathbf{y} = [\, V_a \;\; I_b \,]^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{I}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

19. Write the direct Lyapunov stability criterion for discrete-time nonlinear systems.
Consider the nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.
1) If ...

20. Which of the following functions $V(x_1, x_2)$ are positive definite in the vicinity of the origin?

- ◯ $V(x_1, x_2) = x_1^2(1 - x_1^2) + x_2^2(1 - x_2^2)$;
- ◯ $V(x_1, x_2) = x_1^2 + x_2^2 + x_1 + x_2$;
- ◯ $V(x_1, x_2) = x_1^2 \sin(x_2) + x_2^2 \sin(x_1)$;
- ◯ $V(x_1, x_2) = x_1^2 \cos(x_2) + x_2^2 \cos(x_1)$;

21. Given the following nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x})$, continuous-time and autonomous:

$$\begin{cases} \dot{x}_1 &= \alpha\, x_1 - x_1^3 \\ \dot{x}_2 &= x_1^4 - x_2^3 + \beta\, x_3 \\ \dot{x}_3 &= -\beta\, x_2 - x_3^3 \end{cases}$$

It is easy to verify that the origin $\mathbf{x}_0 = (0,\, 0,\, 0) = 0$ is an equilibrium point for the system.

a) Compute, as a function of parameters α and β, the Jacobian $\mathbf{A}(\mathbf{x})$ of the nonlinear system:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \left[\quad \right]$$

b) Compute, as a function of α and β, the matrix $\mathbf{A}_0$ of the linearized system at point $\mathbf{x}_0 = 0$:

$$\mathbf{A}_0 = \left[\quad \right]$$

c) Study, for varying α and β, the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = 0$ using the reduced Lyapunov criterion:

d) For $\alpha = 0$, study for varying parameter β the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = 0$ using the "direct" Lyapunov criterion and the function: $V(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2$. Eventually, use the La Salle - Krasowskii criterion.

22. Compute the 2 equilibrium points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ of the following *discrete-time* nonlinear system:

$$\begin{cases} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_1(k) + x_2(k)(x_1(k) + 3) \end{cases} \quad \Rightarrow \quad \begin{cases} \tilde{\mathbf{x}}_1 &= (\quad ,\quad) \\ \tilde{\mathbf{x}}_2 &= (\quad ,\quad) \end{cases}$$