System and Control Theory
Test of January 8, 2015
Questions and Exercises

Name:	
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Signature:	

1. Write the discrete time behavior of the output function $\mathbf{y}(t)$, solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and the static equation $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$ at time t_0:

   $$\mathbf{y}(t) =$$

2. Write the explicit form of the *transition matrix* $\mathbf{\Phi}(k,h)$ of the linear time-variant system $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$:

   $$\mathbf{\Phi}(k,h) =$$

3. Describe the meaning of the symbol $\mathcal{X}^-(t_0, t_1, \mathbf{x}(t_1))$:

   $\mathcal{X}^-(t_0, t_1, \mathbf{x}(t_1))$ ànd...

4. Describe the meaning, for discrete linear systems, of the symbol $\mathcal{E}^-(k)$:

   $\mathcal{E}^-(k)$ is ...

   Moreover, write the usual way of computing the set $\mathcal{E}^-(k)$:

   $$\mathcal{E}^-(k) =$$

5. Applying to the dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ the state space transformation $\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}}$ one obtains a transformed system $\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t)$, $\mathbf{y}(t) = \tilde{\mathbf{C}}\,\tilde{\mathbf{x}}(t)$ characterized by the following matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$:

   $$\tilde{\mathbf{A}} = \qquad\qquad \tilde{\mathbf{B}} = \qquad\qquad \tilde{\mathbf{C}} =$$

   The given system and the transformed system satisfy the following properties:

   ○ they have the same eigenvalues;
   ○ $\mathcal{X}^+$ and $\tilde{\mathcal{X}}^+$ have the same dimension;
   ○ they have the same observability matrix;
   ○ they have the same inputs;
   ○ they have the same eigenvectors;
   ○ they have the same transfer matrix;

6. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

   $$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

   The system is: ○ reachable? ○ not-reachable? ○ observable? ○ not-observable?

7. Apply the $\mathscr{Z}$transform to the following *state* function:

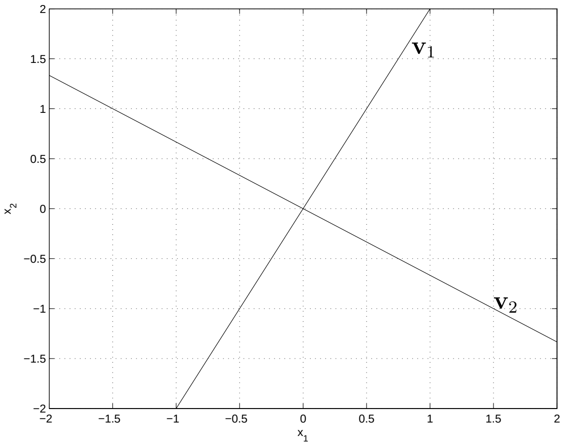
$$\mathscr{Z}\left[\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+\mathbf{B}\mathbf{u}(k)\right]$$

and provides the expression of the transformed function $\mathbf{x}(z)$ of the state vector $\mathbf{x}(k)$ as a function of the initial state $\mathbf{x}_0$ and the transform $\mathbf{u}(z)$ of the input signal $u(k)$:

$$\mathbf{x}(z)=$$

8. Draw qualitatively the trajectories of a second order dynamic system $\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)$ characterized by the eigenvalues λ_i and the eigenvectors $\mathbf{v}_i$ shown in the two following boxes.

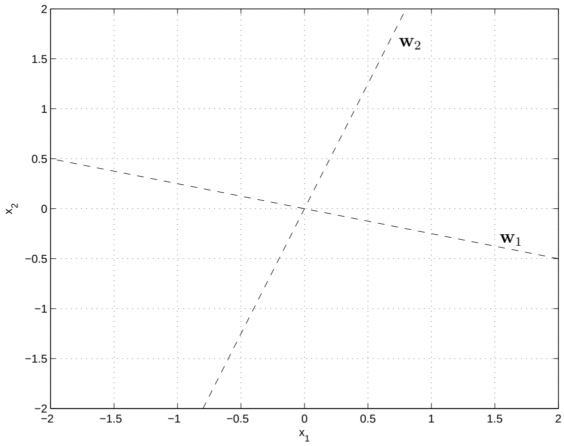
1) Eigenvalues: $\lambda_1=-3$ and $\lambda_2=0$. The corresponding real eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are shown in the figure.



○ Node? ○ Focus? ○ Saddle?

○ Stable? ○ Unstable?

2) Eigenvalues: $\lambda_{1,2}=-1\pm 2j$. Eigenvectors: $\mathbf{v}_1=\mathbf{w}_1+\mathbf{w}_2 j$ and $\mathbf{v}_2=\mathbf{v}_1^*$. The real vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are shown in the figure.



○ Node? ○ Focus? ○ Saddle?

○ Stable? ○ Unstable?

9. Given an autonomous system $\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)$ of the forth order where the matrix $\mathbf{A}$ is characterized by the following eigenvalues λ_i, eigenvectors $\mathbf{v}_i$, and generalized eigenvectors $\mathbf{v}_i^*$:

$$\mathbf{A}=\begin{bmatrix} 5.\overline{6} & 8 & 6.\overline{6} & -1.\overline{3} \\ -3 & -3 & -2 & 0 \\ -0.\overline{3} & -2 & -2.\overline{3} & 0.\overline{6} \\ 5.\overline{6} & 6 & 6.\overline{6} & -0.\overline{3} \end{bmatrix} \quad \begin{array}{l} \lambda_1=1+2j \\ \lambda_2=1-2j \\ \lambda_3=-1 \\ \lambda_4=-1 \end{array} \quad \mathbf{v}_1=\begin{bmatrix} 2j \\ -1-j \\ 1 \\ 2+j \end{bmatrix},\quad \mathbf{v}_1=\begin{bmatrix} -2j \\ -1+j \\ 1 \\ 2-j \end{bmatrix},\quad \mathbf{v}_3=\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix},\quad \mathbf{v}_4^*=\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

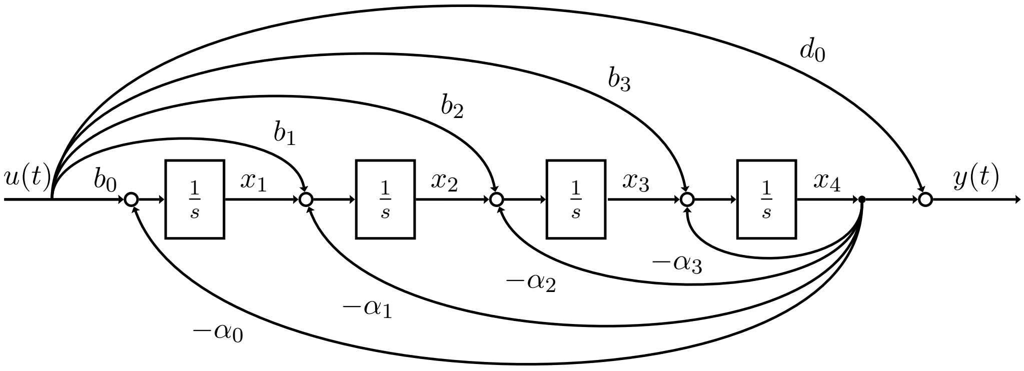
a) Write a transformation matrix $\mathbf{T}$ (with $\mathbf{x}=\mathbf{T}\overline{\mathbf{x}}$) that brings the matrix $\mathbf{A}$ in the Jordan diagonal form $\mathbf{A}_J$:

$$\mathbf{T}=\begin{bmatrix} \qquad\qquad\qquad \\ \\ \\ \end{bmatrix},\qquad \mathbf{A}_J=\begin{bmatrix} \qquad\qquad\qquad \\ \\ \\ \end{bmatrix}$$

b) Write a transformation matrix $\mathbf{T}_R$ (with $\mathbf{x}=\mathbf{T}_R\overline{\mathbf{x}}$) that brings the matrix $\mathbf{A}$ in the “real” Jordan form $\mathbf{A}_R$:

$$\mathbf{T}_R=\begin{bmatrix} \qquad\qquad\qquad \\ \\ \\ \end{bmatrix},\qquad \mathbf{A}_R=\begin{bmatrix} \qquad\qquad\qquad \\ \\ \\ \end{bmatrix}$$

10. Given following block scheme:



Set $\mathbf{x}_c = \left[\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}\right]^{\mathrm{T}}$, write the structure of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of a continuous-time system that, in the state space, describes the dynamics of the given block scheme.

$$\begin{cases} \dot{\mathbf{x}}_c(t) = \left[\begin{array}{c} \qquad\qquad\qquad\qquad \\ \\ \\ \\ \end{array}\right] \mathbf{x}_c(t) + \left[\begin{array}{c} \qquad \\ \\ \\ \\ \end{array}\right] u(t) \\ y(t) = \left[\begin{array}{c} \qquad\qquad\qquad\qquad \end{array}\right] \mathbf{x}_c(t) + \left[\begin{array}{c} \qquad \end{array}\right] u(t) \end{cases}$$

Moreover, compute the transfer function of the system:

$$G(s) = \frac{Y(s)}{U(s)} = \dots$$

11. Given the following nonlinear differential equation:

$$\dddot{y}(t)\, y(t) + 2\, \sin[\ddot{y}(t)] + 3\, \dot{y}^2(t)\, y(t) = u(t).$$

Chosen $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^{\mathrm{T}} = [y(t) \;\; \dot{y}(t) \;\; \ddot{y}(t)]^{\mathrm{T}}$ as state vector, write the corresponding nonlinear differential equation in the state space:

$$\begin{cases} \dot{x}_1 = \\ \dot{x}_2 = \\ \dot{x}_3 = \end{cases}$$

12. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ which move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the solution $\mathbf{u}$ which minimizes the Euclidean norm:

$$\mathbf{u} =$$

13. Given the transfer function $G(z)$, write the structure of the corresponding dynamic system in the reachability canonical form denoting with $u(k)$ the input and with $y(k)$ the output:

$$G(z) = \frac{2z^3 + 6z^2 + 1}{z^4 + 3z^3 + 5z^2 + 2z + 4} + 2 \qquad \left\{ \begin{array}{rcl} \mathbf{x}(k{+}1) & = & \left[\begin{matrix} \qquad\qquad\qquad\qquad \\ \\ \\ \\ \end{matrix} \right] \mathbf{x}(k) + \left[\begin{matrix} \quad \\ \\ \\ \\ \end{matrix} \right] u(k) \\ y(k) & = & \left[\qquad\qquad\qquad\qquad \right] \mathbf{x}(k) + [\quad] u(k) \end{array} \right.$$

14. Compute the following matrix function:

$$e^{\left[\begin{matrix} \sigma & -\omega \\ \omega & \sigma \end{matrix} \right] t} = \left[\begin{matrix} \qquad\qquad\qquad \\ \\ \end{matrix} \right]$$

15. Write the explicit form of the Ackermann formula which provides il vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of a feedback system:

$$\mathbf{k}^{\mathrm{T}} =$$

16. Write the structure of the matrix $\mathbf{P}^{-1}$ of the state space transformation $\mathbf{x} = \mathbf{P}\overline{\mathbf{x}}$ which brings a not-observable system in the standard observability form:

$$\mathbf{P}^{-1} =$$

Moreover, write the block structure of the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$:

$$\overline{\mathbf{A}} = \left[\begin{matrix} \qquad\qquad\qquad \\ \\ \\ \end{matrix} \right], \qquad \overline{\mathbf{B}} = \left[\begin{matrix} \quad \\ \\ \\ \end{matrix} \right], \qquad \overline{\mathbf{C}} = \left[\qquad\qquad\qquad \right]$$

Write the simplified form of transfer matrix $\mathbf{H}(s)$ of the system $\mathcal{S}$ as a function of the sub-matrices $\mathbf{A}_{i,j}$, $\mathbf{B}_i$ and $\mathbf{C}_j$ that characterize the system $\bar{\mathcal{S}} = (\overline{\mathbf{A}},\ \overline{\mathbf{B}},\ \overline{\mathbf{C}})$:

$$\mathbf{H}(s) =$$

17. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of:

a) a *full order closed loop* state estimator:

$$\dot{\hat{\mathbf{x}}}(t) =$$

b) a *reducedl order closed loop* state estimator:

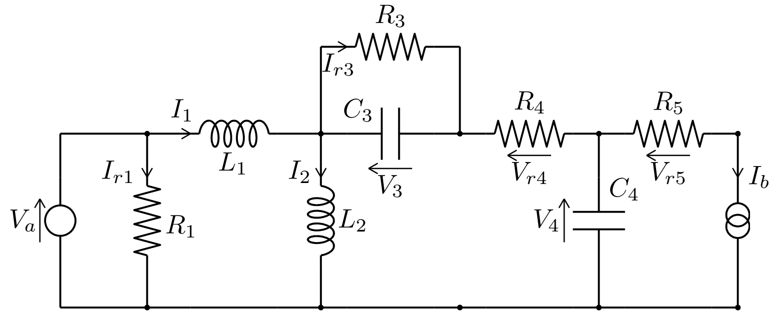
$$\hat{\mathbf{x}}(t) = \mathbf{T} \left[\begin{matrix} \qquad\qquad\qquad \\ \\ \end{matrix} \right]$$

$$\dot{\hat{\mathbf{v}}}(t) =$$

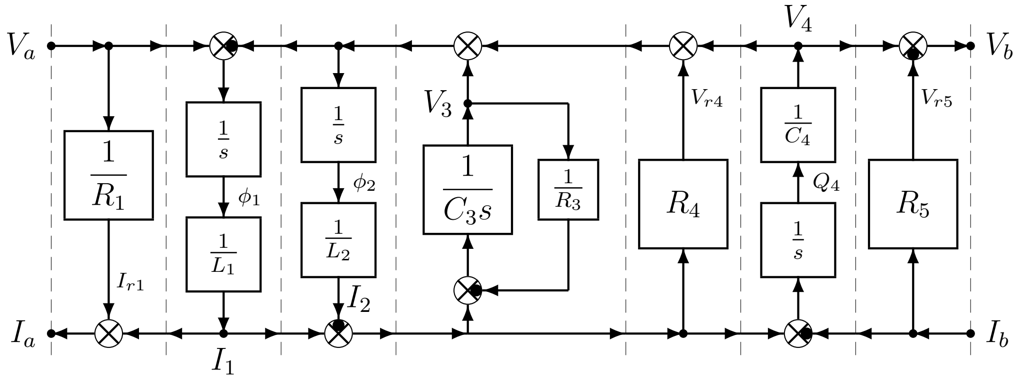
18. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Hydraulic*. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols / Names	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$ q_1 v_1				
$\mathcal{D}_2$ q_2 v_2				
$\mathcal{R}$				

19. Consider the following electric circuit composed by the inductances L_1, L_2, the capacities C_3, C_4 and the resistances R_1, R_3 , R_4 and R_5. Two inputs act on the system: the voltage V_a and the current I_b. The outputs of the system are: the current I_a and the voltage V_b.



The POG model of the given electric circuit has the following structure:



Let $\mathbf{x} = \begin{bmatrix} I_1 & I_2 & V_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

20. Write the instability Lyapunov criterion for continuous-time systems.

Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$*. If:*

1) . . .
2) . . .
3) . . .

then . . .

21. Given the following nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$, *discrete-time and autonomous:*

$$\begin{cases} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= \alpha x_1^2(k) - x_1^3(k) + x_2(k) \end{cases}$$

a) Compute the position of the 2 equilibrium points $\overline{\mathbf{x}}_1$ *and* $\overline{\mathbf{x}}_2$ *of the system:*

$$\overline{\mathbf{x}}_1 = (\quad,\quad), \qquad \overline{\mathbf{x}}_2 = (\quad,\quad),$$

b) Compute the Jacobian $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ *of the nonlinear system* $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$*:*

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

c) Compute the matrices $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *of the linearized system in the neighborhood of the 2 equilibrium points* $\overline{\mathbf{x}}_1$ *and* $\overline{\mathbf{x}}_2$*:*

$$\mathbf{A}_1 = \mathbf{A}(\mathbf{x}_1) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \qquad \mathbf{A}_2 = \mathbf{A}(\mathbf{x}_2) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

d) Study, for varying parameter α, *the stability of the nonlinear system in the neighborhood of the 2 equilibrium points* $\overline{\mathbf{x}}_1$ *e* $\overline{\mathbf{x}}_2$ *using the reduced Lyapunov criterion:*

e) Let be given the function $V(\mathbf{x}(k)) = x_1^2 + x_2^2$. *For* $\alpha = 0$, *compute the function* $\Delta V(\mathbf{x}(k))$ *used in the direct Lyapunov criterion. Do not discuss the obtained final result.*

$$\Delta V(\mathbf{x}(k)) =$$