System and Control Theory
Test of January 8, 2015
Questions and Exercises

Name:	
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1. Write the discrete time behavior of the output function $\mathbf{y}(t)$, solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and the static equation $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$ at time t_0:

$$\mathbf{y}(t) = \mathbf{C}\, e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \mathbf{C}\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

2. Write the explicit form of the *transition matrix* $\mathbf{\Phi}(k,h)$ of the linear time-variant system $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$:

$$\mathbf{\Phi}(k,h) = \begin{cases} \mathbf{A}(k-1)\dots\mathbf{A}(h+1)\mathbf{A}(h) & \text{if } k > h \\ \mathbf{I} \text{ (Matrice identità)} & \text{if } k = h \end{cases}$$

3. Describe the meaning of the symbol $\mathcal{X}^-(t_0, t_1, \mathbf{x}(t_1))$:

   $\mathcal{X}^-(t_0, t_1, \mathbf{x}(t_1))$ is the *set of the states controllable* to the event $\{t_1, \mathbf{x}(t_1)\}$ from the time t_0.

4. Describe the meaning, for discrete linear systems, of the symbol $\mathcal{E}^-(k)$:

   $\mathcal{E}^-(k)$ *is the set of the states* not-observable in k steps, *that is the states compatible with the zero input and output sequences,* $\mathbf{u}(\tau) = 0$ *and* $\mathbf{y}(\tau) = 0$, *for* $\tau \in [0,\ k-1]$.

   Moreover, write the usual way of computing the set $\mathcal{E}^-(k)$:

$$\mathcal{E}^-(k) = \ker \mathcal{O}^-(k) = \ker \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{k-1} \end{bmatrix}$$

5. Applying to the dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ the state space transformation $\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}}$ one obtains a transformed system $\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t)$, $\mathbf{y}(t) = \tilde{\mathbf{C}}\,\tilde{\mathbf{x}}(t)$ characterized by the following matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$:

$$\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \qquad \tilde{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}, \qquad \tilde{\mathbf{C}} = \mathbf{C}\mathbf{T}$$

   The given system and the transformed system satisfy the following properties:

   ⊗ they have the same eigenvalues; ⊗ they have the same inputs;
   ⊗ $\mathcal{X}^+$ and $\tilde{\mathcal{X}}^+$ have the same dimension; ○ they have the same eigenvectors;
   ○ they have the same observability matrix; ⊗ they have the same transfer matrix;

6. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

   The system is: ⊗ reachable? ○ not-reachable? ⊗ observable? ○ not-observable?

7. Apply the $\mathcal{Z}$transform to the following *state* function:

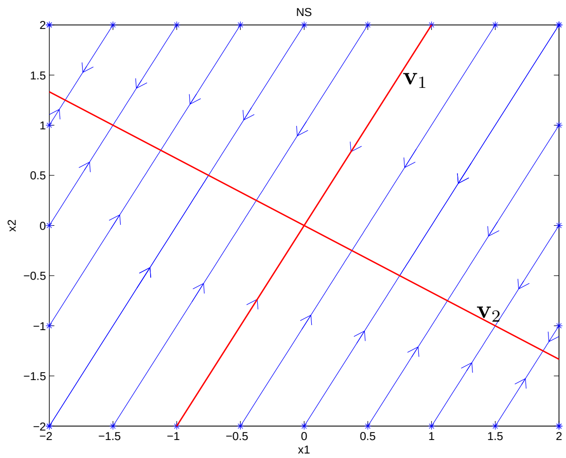
$$\mathcal{Z}\left[\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+\mathbf{B}\mathbf{u}(k)\right]$$

and provides the expression of the transformed function $\mathbf{x}(z)$ of the state vector $\mathbf{x}(k)$ as a function of the initial state $\mathbf{x}_0$ and the transform $\mathbf{u}(z)$ of the input signal $u(k)$:

$$\mathbf{x}(z)=(z\mathbf{I}-\mathbf{A})^{-1}z\,\mathbf{x}_0+(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\,\mathbf{u}(z)$$

8. Draw qualitatively the trajectories of a second order dynamic system $\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)$ characterized by the eigenvalues λ_i and the eigenvectors $\mathbf{v}_i$ shown in the two following boxes.

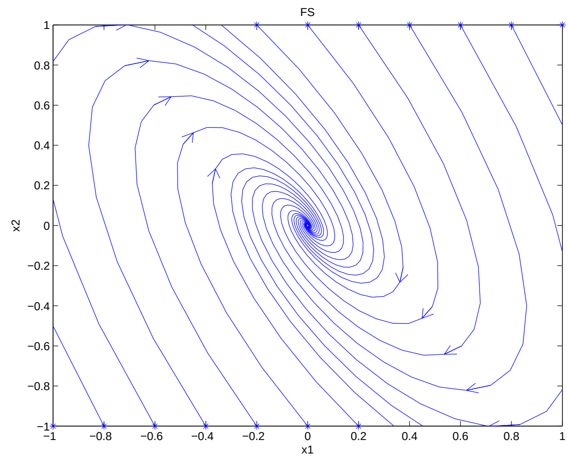
1) Eigenvalues: $\lambda_1=-3$ and $\lambda_2=0$. The corresponding real eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are shown in the figure.



◯ Node? ◯ Focus? ◯ Saddle?

⊗ Stable? ◯ Unstable?

2) Eigenvalues: $\lambda_{1,2}=-1\pm 2j$. Eigenvectors: $\mathbf{v}_1=\mathbf{w}_1+\mathbf{w}_2 j$ and $\mathbf{v}_2=\mathbf{v}_1^*$. The real vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are shown in the figure.



◯ Node? ⊗ Focus? ◯ Saddle?

⊗ Stable? ◯ Unstable?

9. Given an autonomous system $\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)$ of the forth order where the matrix $\mathbf{A}$ is characterized by the following eigenvalues λ_i, eigenvectors $\mathbf{v}_i$, and generalized eigenvectors $\mathbf{v}_i^*$:

$$\mathbf{A}=\begin{bmatrix}5.\bar{6} & 8 & 6.\bar{6} & -1.\bar{3}\\ -3 & -3 & -2 & 0\\ -0.\bar{3} & -2 & -2.\bar{3} & 0.\bar{6}\\ 5.\bar{6} & 6 & 6.\bar{6} & -0.\bar{3}\end{bmatrix}\quad \begin{array}{l}\lambda_1=1+2j\\ \lambda_2=1-2j\\ \lambda_3=-1\\ \lambda_4=-1\end{array}\quad \mathbf{v}_1=\begin{bmatrix}2j\\ -1-j\\ 1\\ 2+j\end{bmatrix},\quad \mathbf{v}_1=\begin{bmatrix}-2j\\ -1+j\\ 1\\ 2-j\end{bmatrix},\quad \mathbf{v}_3=\begin{bmatrix}0\\ -1\\ 1\\ -1\end{bmatrix},\quad \mathbf{v}_4^*=\begin{bmatrix}1\\ 0\\ -1\\ 0\end{bmatrix}.$$

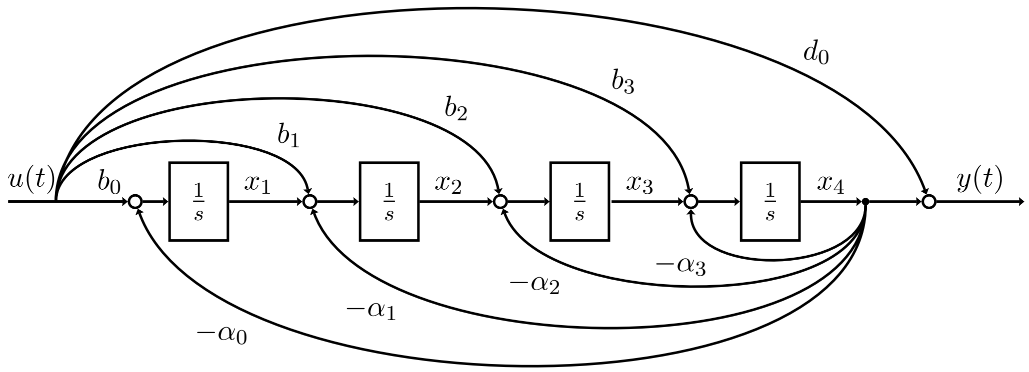
a) Write a transformation matrix $\mathbf{T}$ (with $\mathbf{x}=\mathbf{T}\bar{\mathbf{x}}$) that brings the matrix $\mathbf{A}$ in the Jordan diagonal form $\mathbf{A}_J$:

$$\mathbf{T}=\begin{bmatrix}2j & -2j & 0 & 1\\ -1-j & -1+j & -1 & 0\\ 1 & 1 & 1 & -1\\ 2+j & 2-j & -1 & 0\end{bmatrix},\qquad \mathbf{A}_J=\begin{bmatrix}1+2j & 0 & 0 & 0\\ 0 & 1-2j & 0 & 0\\ 0 & 0 & -1 & 1\\ 0 & 0 & 0 & -1\end{bmatrix}$$

b) Write a transformation matrix $\mathbf{T}_R$ (with $\mathbf{x}=\mathbf{T}_R\bar{\mathbf{x}}$) that brings the matrix $\mathbf{A}$ in the "real" Jordan form $\mathbf{A}_R$:

$$\mathbf{T}_R=\begin{bmatrix}0 & 2 & 0 & 1\\ -1 & -1 & -1 & 0\\ 1 & 0 & 1 & -1\\ 2 & 1 & -1 & 0\end{bmatrix},\qquad \mathbf{A}_R=\begin{bmatrix}1 & 2 & 0 & 0\\ -2 & 1 & 0 & 0\\ 0 & 0 & -1 & 1\\ 0 & 0 & 0 & -1\end{bmatrix}$$

10. Given following block scheme:



Set $\mathbf{x}_c = \left[\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}\right]^{\mathrm{T}}$, write the structure of the matrices **A**, **B** and **C** of a continuous-time system that, in the state space, describes the dynamics of the given block scheme.

$$\left\{\begin{array}{rcl} \dot{\mathbf{x}}_o(t) & = & \left[\begin{array}{cccc} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] + \left[\begin{array}{c} b_0 \\ b_1 \\ b_2 \\ b_3 \end{array}\right] u(t) \\ y(t) & = & \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right] \mathbf{x}_o(t) \;+\; d_0\, u(t) \end{array}\right.$$

Moreover, compute the transfer function of the system:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3\, s^3 + b_2\, s^2 + b_1\, s + b_0}{s^4 + \alpha_3\, s^3 + \alpha_2\, s^2 + \alpha_1\, s + \alpha_0} + d_0$$

11. Given the following nonlinear differential equation:

$$\dddot{y}(t)\, y(t) + 2\, \sin[\ddot{y}(t)] + 3\, \dot{y}^2(t)\, y(t) = u(t).$$

Chosen $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^{\mathrm{T}} = [y(t) \;\; \dot{y}(t) \;\; \ddot{y}(t)]^{\mathrm{T}}$ as state vector, write the corresponding nonlinear differential equation in the state space:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & \dfrac{u(t) - 2\, \sin x_3}{x_1} - 3\, x_2^2 \end{array}\right.$$

12. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions **u** which move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\; k]$ write the solution **u** which minimizes the Euclidean norm:

$$\mathbf{u} = (\mathcal{R}_k^+)^T [\mathcal{R}_k^+ (\mathcal{R}_k^+)^T]^{-1} [\mathbf{x}(k) - \mathbf{A}^k \mathbf{x}(0)]$$

13. Given the transfer function $G(z)$, write the structure of the corresponding dynamic system in the reachability canonical form denoting with $u(k)$ the input and with $y(k)$ the output:

$$G(z) = \frac{2z^3 + 6z^2 + 1}{z^4 + 3z^3 + 5z^2 + 2z + 4} + 2$$

$$\left\{\begin{array}{l} \mathbf{x}(k{+}1) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -2 & -5 & -3 \end{array}\right] \mathbf{x}(k) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right] u(k) \\ \\ y(k) = \left[\begin{array}{cccc} 1 & 0 & 6 & 2 \end{array}\right] \mathbf{x}(k) + \left[\begin{array}{c} 2 \end{array}\right] u(k) \end{array}\right.$$

14. Compute the following matrix function:

$$e^{\begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} t} = \begin{bmatrix} e^{\sigma t}\cos(\omega t) & -e^{\sigma t}\sin(\omega t) \\ e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) \end{bmatrix}$$

15. Write the explicit form of the Ackermann formula which provides il vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of a feedback system:

$$\mathbf{k}^{\mathrm{T}} = -\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} (\mathcal{R}^{+})^{-1} p(\mathbf{A})$$

16. Write the structure of the matrix $\mathbf{P}^{-1}$ of the state space transformation $\mathbf{x} = \mathbf{P}\overline{\mathbf{x}}$ which brings a not-observable system in the standard observability form:

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} \qquad \text{where} \qquad \mathrm{Im}\mathbf{P}_1^{\mathrm{T}} = \mathrm{Im}(\mathcal{O}^{-})^{\mathrm{T}} \text{ and } \mathbf{P}_2 \text{ makes non singular the matrix } \mathbf{P}^{-1}.$$

Moreover, write the block structure of the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$:

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{1,1} & 0 \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix} \qquad \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & 0 \end{bmatrix}$$

Write the simplified form of transfer matrix $\mathbf{H}(s)$ of the system $\mathcal{S}$ as a function of the sub-matrices $\mathbf{A}_{i,j}$, $\mathbf{B}_i$ and $\mathbf{C}_j$ that characterize the system $\bar{\mathcal{S}} = (\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\mathbf{C}})$:

$$\mathbf{H}(s) = \mathbf{C}_1(s\,\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{B}_1$$

17. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of:

a) a *full order closed loop* state estimator:

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} + \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t)$$

b) a *reducedl order closed loop* state estimator:

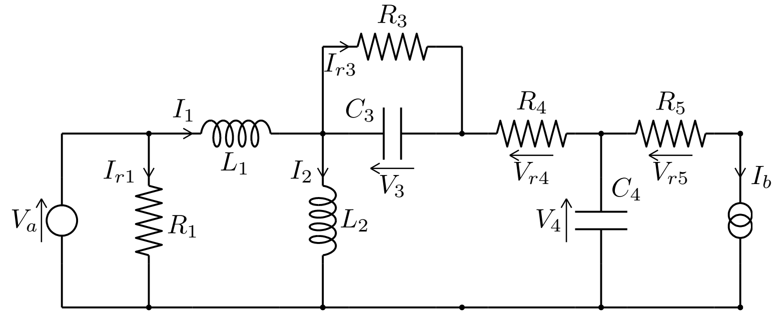
$$\hat{\mathbf{x}}(t) = \mathbf{T}\begin{bmatrix} \hat{\mathbf{v}}(t) - \mathbf{L}\mathbf{y}(t) \\ \mathbf{y}(t) \end{bmatrix}$$

$$\dot{\hat{\mathbf{v}}}(t) = (\overline{\mathbf{A}}_{11} + \mathbf{L}\overline{\mathbf{A}}_{21})\hat{\mathbf{v}}(t) + (\overline{\mathbf{A}}_{12} + \mathbf{L}\overline{\mathbf{A}}_{22} - \overline{\mathbf{A}}_{11}\mathbf{L} - \mathbf{L}\overline{\mathbf{A}}_{21}\mathbf{L})\mathbf{y}(t) + (\mathbf{B}_1 + \mathbf{L}\mathbf{B}_2)\mathbf{u}(t)$$

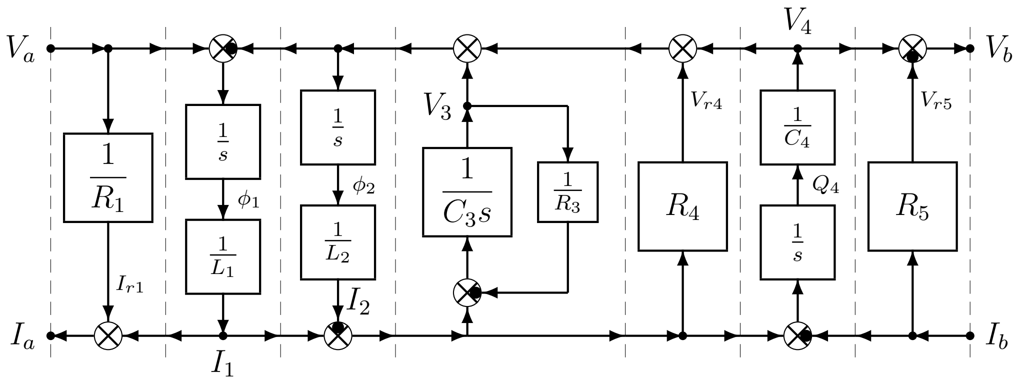
18. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Hydraulic*. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$ q_1 v_1	C_I Hydr. Capacity V Volume P Pressure	$V = \Phi_C(P)$	$V = C_I\,P$	$\frac{dV}{dt} = Q$
$\mathcal{D}_2$ q_2 v_2	L_I Hydr. Inductance ϕ_I Hydr. Flux Q Volume flow rate	$\phi_I = \Phi_L(Q)$	$\phi_I = L_I\,Q$	$\frac{d\phi_I}{dt} = P$
$\mathcal{R}$	R Hydr. Resistance	$P = \Phi_R(Q)$	$P = R_I\,Q$	

19. Consider the following electric circuit composed by the inductances L_1, L_2, the capacities C_3, C_4 and the resistances R_1, R_3 , R_4 and R_5. Two inputs act on the system: the voltage V_a and the current I_b. The outputs of the system are: the current I_a and the voltage V_b.



The POG model of the given electric circuit has the following structure:



Let $\mathbf{x} = \begin{bmatrix} I_1 & I_2 & V_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_4 & R_4 & -1 & -1 \\ R_4 & -R_4 & 1 & 1 \\ 1 & -1 & -\frac{1}{R_3} & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & -R_5 \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

20. Write the <u>instability</u> Lyapunov criterion for continuous-time systems.
Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$*. If:*

1) in a neighborhood W of $\mathbf{x}_0$ <u>it exists</u> a continuous function $V(\mathbf{x}) : W \to \mathcal{R}$ with continuous first time-derivatives and zero in $\mathbf{x}_0$;
2) the point $\mathbf{x}_0$ is an accumulation point for the set of points $\mathbf{x} \in W$ in which it is $V(\mathbf{x}) > 0$;
3) $\dot{V}(\mathbf{x})$ is *positive definite* in W;

then $\mathbf{x}_0$ is an <u>*unstable*</u> equilibrium point.

21. Given the following nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$, discrete-time and autonomous:

$$\begin{cases} x_1(k+1) &=& x_2(k) \\ x_2(k+1) &=& \alpha x_1^2(k) - x_1^3(k) + x_2(k) \end{cases}$$

a) Compute the position of the 2 equilibrium points $\overline{\mathbf{x}}_1$ and $\overline{\mathbf{x}}_2$ of the system:

The equilibrium points of the system can be determined imposing $x_1(k+1) = x_1(k)$ and $x_2(k+1) = x_2(k)$:

$$x_1 = x_2, \qquad x_2 = \alpha x_1^2 - x_1^3 + x_2 \quad \Leftrightarrow \quad 0 = (\alpha - x_1)x_1^2.$$

The system has the following 2 equilibrium points:

$$\overline{\mathbf{x}}_1 = (0,\ 0), \qquad \overline{\mathbf{x}}_2 = (\alpha,\ \alpha).$$

b) Compute the Jacobian $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ of the nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$:

The Jacobian of the nonlinear system is:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 2\alpha x_1 - 3x_1^2 & 1 \end{bmatrix}$$

c) Compute the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of the linearized system in the neighborhood of the 2 equilibrium points $\overline{\mathbf{x}}_1$ and $\overline{\mathbf{x}}_2$:

The matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of the linearized system have the following structure:

$$\mathbf{A}_1 = \mathbf{A}(\mathbf{x}_1) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{A}_2 = \mathbf{A}(\mathbf{x}_2) = \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 1 \end{bmatrix},$$

d) Study, for varying parameter α, the stability of the nonlinear system in the neighborhood of the 2 equilibrium points $\overline{\mathbf{x}}_1$ e $\overline{\mathbf{x}}_2$ using the reduced Lyapunov criterion:

The characteristic polynomial of matrix $\mathbf{A}_1$ is the following:

$$\Delta_{\mathbf{A}_1}(z) = z(z-1) = 0 \qquad \rightarrow \qquad z_1 = 0, \quad z_2 = 1.$$

The second eigenvalue $z_2 = 1$ of the linearized system is located on the unitary circle and therefore in this case the reduced Lyapunov criterion cannot be used.

The characteristic polynomial of matrix $\mathbf{A}_2$ is the following:

$$\Delta_{\mathbf{A}_2}(z) = z^2 - z + \alpha^2 = 0 \qquad \rightarrow \qquad z_{1,2} = 0.5 \pm \sqrt{0.25 - \alpha^2}.$$

The reduced Lyapunov criterion cannot be used for $|z_{1,2}| = 1$, that is for:

$$\alpha = 0 \qquad \alpha^2 = 1.$$

Using the reduced Lyapunov criterion it can be stated that the equilibrium point $\mathbf{x}_2 = (\alpha, \alpha)$ is asymptotically stable for the given nonlinear system for $|z_{1,2}| < 1$, that is for $|\alpha| < 1$. The point $\mathbf{x}_2$ is unstable for $|z_{1,2}| > 1$, that is for $|\alpha| > 1$.

e) Let be given the function $V(\mathbf{x}(k)) = x_1^2 + x_2^2$. For $\alpha = 0$, compute the function $\Delta V(\mathbf{x}(k))$ used in the direct Lyapunov criterion. Do not discuss the obtained final result.

$$\Delta V(\mathbf{x}(k)) = (x_2)^2 + (x_2 - x_1^3)^2 - x_1^2 - x_2^2 = (x_2 - x_1^3)^2 - x_1^2$$