System and Control Theory
Test of February 3, 2014
Questions and Exercises

Name:	
Nr. Mat.	
Signature:	

1. Write the explicit solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ being $\mathbf{x}(h)$ the initial state at time h.

$$\mathbf{x}(k) =$$

2. Write the explicit form of the *transition matrix* $\mathbf{\Phi}(t_0, t)$ of a continuous-time, time-invariant linear systems:

$$\mathbf{\Phi}(t, t_0) =$$

3. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \quad \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \quad \end{bmatrix}$$

The system is: ◯ reachable ◯ not-reachable ◯ observable ◯ not-observable

Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \mathrm{Im}\,[\mathcal{B}_R] = \mathrm{Im}\begin{bmatrix} \quad \end{bmatrix}, \qquad \mathcal{E}^- = \mathrm{Im}\,[\mathcal{B}_O] = \mathrm{Im}\begin{bmatrix} \quad \end{bmatrix}.$$

4. Write the formula for computing the state transition matrix of the system $\mathbf{A}^k$ of a discrete-time linear system using the $\mathcal{Z}$-transform:

$$\mathbf{A}^k =$$

5. The *algebraic* multiplicity of an eigenvalue λ of matrix $\mathbf{A}$

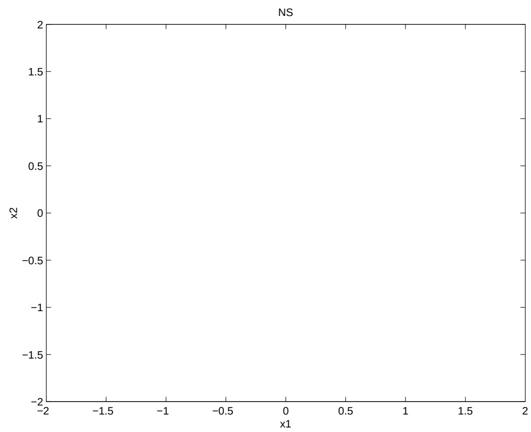
   ◯ is the dimension of the state space U_λ associated to the eigenvalue λ;
   ◯ is the dimension of the Jordan block $\mathbf{J}_\lambda$ associated to the eigenvalue λ;
   ◯ is the multiplicity degree of λ in the characteristic polynomial of matrix $\mathbf{A}$;
   ◯ is the number of eigenvectors linearly independent associated to the eigenvalue λ;

6. Show the structure of the generic Jordan block $\mathbf{J}_i$ and the generic Jordan miniblock $\mathbf{J}_{i,j}$ of the i-th eigenvalue λ_i of of a generic matrix $\mathbf{A}$.

$$\mathbf{J}_i = \begin{bmatrix} \quad \end{bmatrix}, \qquad \mathbf{J}_{i,j} = \begin{bmatrix} \quad \end{bmatrix}$$

7. Considered a **discrete-time** dynamic system $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k)$ of the second order characterized by two complex conjugate eigenvalues $\lambda_1 = -1 + 2\,j$, $\lambda_2 = -1 - 2\,j$, answer the following questions and draw the qualitative behavior of the state trajectories in the vicinity of the origin:

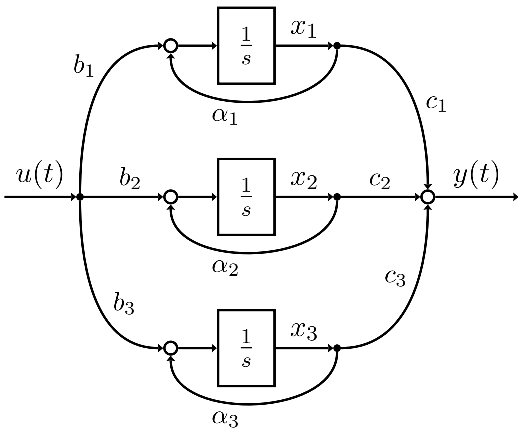
- ◯ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are real and different.
- ◯ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are complex conjugate.
- ◯ for $t \to \infty$ all the trajectories tend to flatten on one of the two eigenvectors.
- ◯ for $t \to \infty$ all the trajectories tend to zero.



Which name is typically used for denoting the type of trajectories shown above:

◯ Node? ◯ Focus? ◯ Saddle? | ◯ Degenerate? ◯ Stable? ◯ Unstable?

8. Given the block scheme shown in figure, write the structure of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of corresponding dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$, $\mathbf{y} = \mathbf{C}\,\mathbf{x}(t)$ when $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}}$.



$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} \qquad\qquad\qquad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \quad \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \qquad\qquad\qquad \end{bmatrix} \mathbf{x}(t) \end{cases}$$

9. A nilpotent matrix $\mathbf{N}$ of order ν is a matrix

- ◯ such that $\mathbf{N}^\nu = \mathbf{I}$;
- ◯ such that $\mathbf{N}^\nu = \mathbf{0}$;
- ◯ that has all the eigenvalues unitari;
- ◯ that has all the eigenvalues equal to 0;

10. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0)]^{\mathrm{T}}$, the free evolution of the continuous-time autonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t)$:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \sigma & -\omega & 0 \\ \omega & \sigma & 0 \\ 0 & 0 & \delta \end{bmatrix} \mathbf{x}(t) \qquad \mathbf{x}(t) = \begin{bmatrix} \qquad\qquad\qquad \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

11. Given the dynamic system shown below, write the transfer function $G(s)$ which links the input $u(t)$ to the output $y(t)$:

$G(s) =$

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 & 0 & -6 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 \\ 5 \\ 1 \\ 3 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

12. Let $\mathbf{A}$ and $\overline{\mathbf{A}}$ be two similar matrices $\mathbf{A} = \mathbf{T}\overline{\mathbf{A}}\mathbf{T}^{-1}$. The matrix function $e^{\mathbf{A}t}$ satisfies the following property:

○ $e^{\mathbf{A}t} = \mathbf{T}^{-1}\, e^{\overline{\mathbf{A}}t}\, \mathbf{T}$; ○ $e^{\mathbf{A}t} = \mathbf{T}\, e^{\overline{\mathbf{A}}t}\, \mathbf{T}^{-1}$; ○ $e^{\mathbf{A}t} = e^{\mathbf{T}}\, e^{\overline{\mathbf{A}}t}\, e^{\mathbf{T}^{-1}}$;

13. Given a SISO linear system completely reachable characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$. Write the structure of the matrices $\mathbf{A}_c$, $\mathbf{b}_c$ and $\mathbf{c}_c$ of the corresponding controllability canonical form. Let $p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_0$ the characteristic polynomial of matrix $\mathbf{A}$.

$$\mathbf{A}_C = \left[\quad \right], \quad \mathbf{b}_C = \left[\quad \right], \quad \mathbf{c}_C = \left[\quad \right]$$

Moreover, write the structure of matrix $\mathbf{T}$ that (using state space transformation $\mathbf{x} = \mathbf{T}\mathbf{x}_c$) brings the system in the controllability canonical form.

$$\mathbf{T} =$$

14. Give the meaning of the symbol $\mathcal{X}^+(t_0, t_1, \mathbf{x}(t_0))$:

...

15. Give the meaning of the symbol $\mathcal{E}^-(t_0, t_1, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$:

...

16. Let $\mathcal{S}_D$ be the dual system of the discrete system $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$:

○ If $\mathcal{S}$ is reachable $\Rightarrow \mathcal{S}_D$ is observable;
○ If $\mathcal{S}$ is controllable $\Rightarrow \mathcal{S}_D$ is observable;
○ If $\mathcal{S}$ is observable $\Rightarrow \mathcal{S}_D$ is controllable;
○ If $\mathcal{S}$ is constructable $\Rightarrow \mathcal{S}_D$ is controllable;

17. Given a system $(\mathbf{A},\ \mathbf{b})$ completely reachable. The corresponding sampled system (being T the sampling period) is completely reachable if and only if for each couple λ_i, λ_j of eigenvalues of $\mathbf{A}$ having the same real part, it is:

18. For the discrete time-invariant linear system $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,u(k)$, the input sequence $\mathbf{u}(0), \ldots, \mathbf{u}(\bar{k}-1)$ which moves the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(\bar{k})$:

- ◯ exists if the system is reachable;
- ◯ exists if $\mathbf{x}(\bar{k}) - e^{\mathbf{A}\bar{k}}\mathbf{x}(0) \in \mathcal{X}^+(\bar{k})$;
- ◯ exists if $\mathbf{x}(\bar{k}) \in \mathcal{X}^+(\bar{k})$;
- ◯ exists if $\mathbf{x}(\bar{k}) - \mathbf{A}^{\bar{k}}\mathbf{x}(0) \in \mathcal{X}^+(\bar{k})$;

19. Given the following continuous-time linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

Thinking to the block structure of the systems in standard form, it is possible to state that:

- ◯ the system is in the reachability standard form;
- ◯ the system is in the standard observability form;
- ◯ it is possibile to build a state observer for this system;
- ◯ it is possibile to stabilize the system with a static state feedback;

Using the structural properties of the system compute the transfer function $G(s) = \frac{Y(s)}{U(s)}$ which links the input $U(s) = \mathcal{L}[u(t)]$ to the output $Y(s) = \mathcal{L}[y(t)]$

$$G(s) =$$

20. The Ackermann formula for computing the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of the feedback system can be used

- ◯ for whatever system;
- ◯ only if the system is observable;
- ◯ only if the system is reachable;
- ◯ only for systems with one input;

21. Given the discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, write the structure of:

a) a *full order closed loop* state estimator:

$$\hat{\mathbf{x}}(k+1) =$$

b) a *reducedl order closed loop* state estimator:

$$\hat{\mathbf{x}}(k) = \mathbf{T} \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\hat{\mathbf{v}}(k+1) =$$

22. Write the direct Lyapunov stability criterion for continuous-time systems.

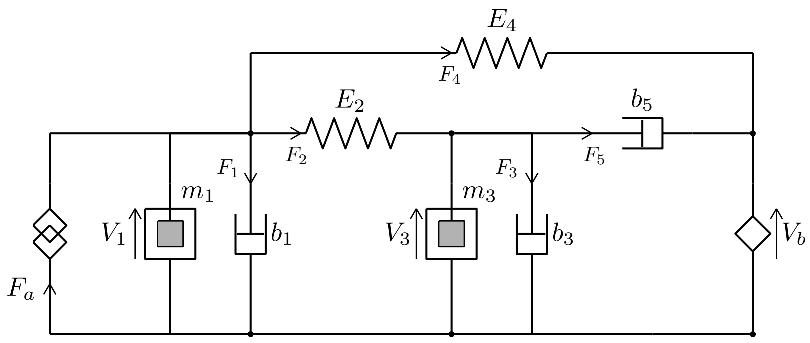
Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.

1) If ...

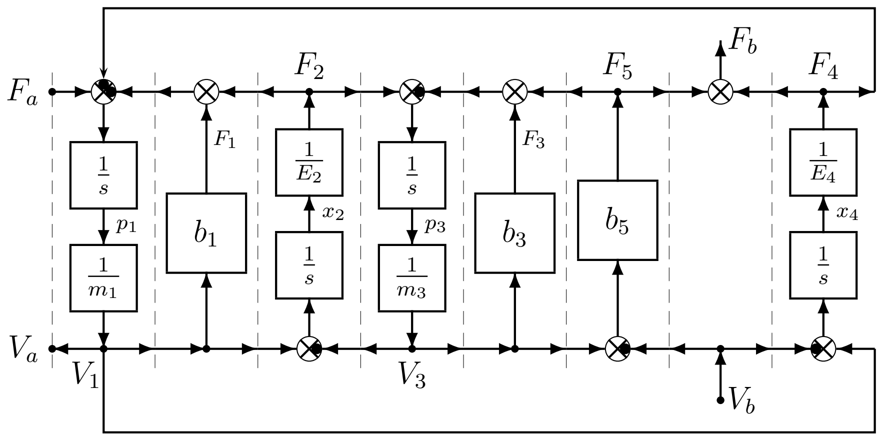
23. Which of the following functions $V(x_1, x_2)$ are positive definite in the vicinity of the origin?

- ○ $V(x_1, x_2) = x_1^2 x_2^2$;
- ○ $V(x_1, x_2) = x_1^2 x_2^2 (x_1^2 + x_2^2)$;
- ○ $V(x_1, x_2) = x_1^2 + x_2^2$;
- ○ $V(x_1, x_2) = x_1^2 + x_2^2 - x_1^4 - x_2^4$;

24. Consider the following mechanical system composed by masses m_1, m_3, elasticities E_2, E_4 and dampers b_1, b_2 and b_5. Two inputs act on the system: the force F_a and the velocity V_b. The outputs of the system are: velocity $V_a = V_1$ and the force $F_b = F_4 + F_5$.



The POG model of the given mechanical system has the following structure:



Let $\mathbf{x} = \begin{bmatrix} V_1 & F_2 & V_3 & F_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} F_a & V_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} V_a & F_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{F}_2 \\ \dot{V}_3 \\ \dot{F}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} V_1 \\ F_2 \\ V_3 \\ F_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} F_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ F_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} F_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

25. Given an electric element characterized by an output voltage V, an input current I and a nonlinear capacity $C(V)$ which is function of the voltage V. Write : 1) the constitutive equation of the electric element; 2) the differential equation that describes the dynamics of the considered physical element:

1) $\ldots$ 2) $\frac{d}{dt}\begin{bmatrix} \quad \end{bmatrix} =$

26. Given the following autonomous discrete-time nonlinear system:

$$\begin{cases} x_1(k+1) &= 0.5 \sin x_2(k) + x_1(k) \\ x_2(k+1) &= 1 - x_1(k) + x_2^2(k) \sin x_2(k) \end{cases}$$

a) Verify if the point $\mathbf{x}_0 = (1, 0)$ is an equilibrium point for the system:

b) Linearize the system in the neighborhood of point $\mathbf{x}_0 = (1, 0)$ computing the matrix $\mathbf{A}$ of the corresponding linearized system:

c) Study the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = (1, 0)$ using the reduced Lyapunov criterion:

27. Given the following autonomous discrete-time nonlinear system:

$$\begin{cases} x_1(k+1) &= x_1(k) + 3\,x_2^2(k) \\ x_2(k+1) &= 3\,x_2^2(k) \end{cases}$$

It is easy to verify that $\mathbf{x}_0 = (0, 0)$ is an equilibrium point per the system. Study the stability of point $\mathbf{x}_0 = (0, 0)$ using the "direct" Lyapunov criterion and the function: $V(\mathbf{x}) = x_1^2 + x_2^2$. Eventually, use the La Salle - Krasowskii criterion.