System and Control Theory
Test of February 3, 2014
Questions and Exercises

Name:	
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1. Write the explicit solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ being $\mathbf{x}(h)$ the initial state at time h.

$$\mathbf{x}(k) = \mathbf{A}^{k-h}\mathbf{x}(h) + \sum_{j=h}^{k-1} \mathbf{A}^{k-j-1}\mathbf{B}\,\mathbf{u}(j)$$

2. Write the explicit form of the *transition matrix* $\mathbf{\Phi}(t_0, t)$ of a continuous-time, time-invariant linear systems:

$$\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)}$$

3. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

The system is: ⊗ reachable ◯ not-reachable ◯ observable ⊗ not-observable

Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \mathrm{Im}\mathcal{R}^+ = \mathrm{Im}\,[\mathcal{B}_R] = \mathrm{Im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathcal{E}^- = \ker\mathcal{O}^- = \mathrm{Im}\,[\mathcal{B}_O] = \mathrm{Im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

4. Write the formula for computing the state transition matrix of the system $\mathbf{A}^k$ of a discrete-time linear system using the $\mathcal{Z}$-transform:

$$\mathbf{A}^k = \mathcal{Z}^{-1}[z(z\mathbf{I} - \mathbf{A})^{-1}]$$

5. The *algebraic* multiplicity of an eigenvalue λ of matrix $\mathbf{A}$

◯ is the dimension of the state space U_λ associated to the eigenvalue λ;
⊗ is the dimension of the Jordan block $\mathbf{J}_\lambda$ associated to the eigenvalue λ;
⊗ is the multiplicity degree of λ in the characteristic polynomial of matrix $\mathbf{A}$;
◯ is the number of eigenvectors linearly independent associated to the eigenvalue λ;

6. Show the structure of the generic Jordan block $\mathbf{J}_i$ and the generic Jordan miniblock $\mathbf{J}_{i,j}$ of the i-th eigenvalue λ_i of of a generic matrix $\mathbf{A}$.

$$\mathbf{J}_i = \begin{bmatrix} \mathbf{J}_{i,1} & 0 & \dots & 0 \\ 0 & \mathbf{J}_{i,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{J}_{i,\nu_i} \end{bmatrix}, \qquad \mathbf{J}_{i,j} = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix}$$

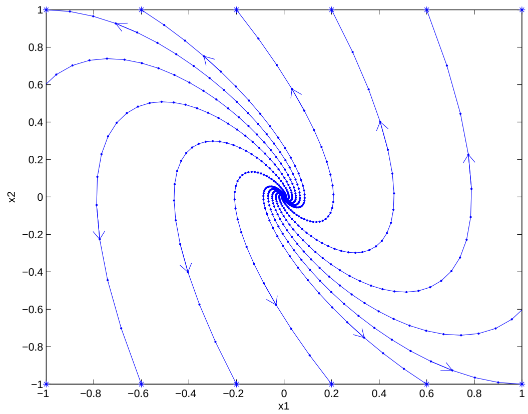
7. Considered a **discrete-time** dynamic system $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k)$ of the second order characterized by two complex conjugate eigenvalues $\lambda_1 = -1 + 2\,j$, $\lambda_2 = -1 - 2\,j$, answer the following questions and draw the qualitative behavior of the state trajectories in the vicinity of the origin:

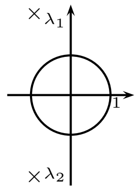
◯ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are real and different.

⊗ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are complex conjugate.

◯ for $t \to \infty$ all the trajectories tend to flatten on one of the two eigenvectors.

◯ for $t \to \infty$ all the trajectories tend to zero.

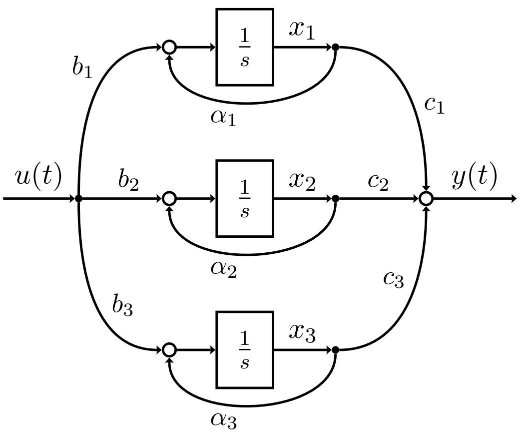




Which name is typically used for denoting the type of trajectories shown above:

◯ Node? ⊗ Focus? ◯ Saddle? | ◯ Degenerate? ◯ Stable? ⊗ Unstable?

8. Given the block scheme shown in figure, write the structure of the matrices **A**, **B** and **C** of corresponding dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$, $\mathbf{y} = \mathbf{C}\,\mathbf{x}(t)$ when $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}}$.



$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

9. A nilpotent matrix $\mathbf{N}$ of order ν is a matrix

◯ such that $\mathbf{N}^\nu = \mathbf{I}$;

◯ that has all the eigenvalues unitari;

⊗ such that $\mathbf{N}^\nu = \mathbf{0}$;

⊗ that has all the eigenvalues equal to 0;

10. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0)]^{\mathrm{T}}$, the free evolution of the continuous-time autonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t)$:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \sigma & -\omega & 0 \\ \omega & \sigma & 0 \\ 0 & 0 & \delta \end{bmatrix} \mathbf{x}(t) \qquad \mathbf{x}(t) = \begin{bmatrix} e^{\sigma t}\cos(\omega t) & -e^{\sigma t}\sin(\omega t) & 0 \\ e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) & 0 \\ 0 & 0 & e^{\delta t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

11. Given the dynamic system shown below, write the transfer function $G(s)$ which links the input $u(t)$ to the output $y(t)$:

$$G(s) = \frac{3s^3 + s^2 + 5s + 2}{s^4 + s^3 + 4s^2 + 3s + 6}$$

$$\begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 0 & 0 & -6 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 \\ 5 \\ 1 \\ 3 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

12. Let $\mathbf{A}$ and $\overline{\mathbf{A}}$ be two similar matrices $\mathbf{A} = \mathbf{T}\overline{\mathbf{A}}\mathbf{T}^{-1}$. The matrix function $e^{\mathbf{A}t}$ satisfies the following property:

○ $e^{\mathbf{A}t} = \mathbf{T}^{-1}\, e^{\overline{\mathbf{A}}t}\, \mathbf{T}$; ⊗ $e^{\mathbf{A}t} = \mathbf{T}\, e^{\overline{\mathbf{A}}t}\, \mathbf{T}^{-1}$; ○ $e^{\mathbf{A}t} = e^{\mathbf{T}}\, e^{\overline{\mathbf{A}}t}\, e^{\mathbf{T}^{-1}}$;

13. Given a SISO linear system completely reachable characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$. Write the structure of the matrices $\mathbf{A}_c$, $\mathbf{b}_c$ and $\mathbf{c}_c$ of the corresponding controllability canonical form. Let $p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_0$ the characteristic polynomial of matrix $\mathbf{A}$.

$$\mathbf{A}_C = \begin{bmatrix} 0 & 1 & 0\ldots & 0 \\ 0 & 0 & 1\ldots & 0 \\ \vdots & \vdots & & \vdots \\ -\alpha_0 & -\alpha_1 & \ldots & -\alpha_{n-1} \end{bmatrix}, \qquad \mathbf{b}_C = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \qquad \mathbf{c}_C = \begin{bmatrix} \beta_0 & \beta_1 & \ldots & \beta_{n-1} \end{bmatrix}$$

Moreover, write the structure of matrix $\mathbf{T}$ that (using state space transformation $\mathbf{x} = \mathbf{T}\mathbf{x}_c$) brings the system in the controllability canonical form.

$$\mathbf{T} = \mathcal{R}^+(\mathcal{R}_c^+)^{-1} = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \ldots & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \ldots & \ldots & 1 & 0 \\ \alpha_3 & \ldots & \ldots & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \ldots & \ldots & 0 & 0 \\ 1 & 0 & \ldots & \ldots & 0 & 0 \end{bmatrix}$$

14. Give the meaning of the symbol $\mathcal{X}^+(t_0, t_1, \mathbf{x}(t_0))$:

It is the set of the reachable states at time t_1 starting from $\{t_0, \mathbf{x}(t_0)\}$.

15. Give the meaning of the symbol $\mathcal{E}^-(t_0, t_1, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$:

It is the set of the initial states $\mathbf{x}(t_0)$ compatible with the input and output functions $\mathbf{u}(\cdot)$ and $\mathbf{y}(\cdot)$ in the time interval $[t_0, t_1]$.

16. Let $\mathcal{S}_D$ be the dual system of the discrete system $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$:

⊗ If $\mathcal{S}$ is reachable $\Rightarrow$ $\mathcal{S}_D$ is observable;
○ If $\mathcal{S}$ is controllable $\Rightarrow$ $\mathcal{S}_D$ is observable;
⊗ If $\mathcal{S}$ is observable $\Rightarrow$ $\mathcal{S}_D$ is controllable;
⊗ If $\mathcal{S}$ is constructable $\Rightarrow$ $\mathcal{S}_D$ is controllable;

17. Given a system $(\mathbf{A},\ \mathbf{b})$ completely reachable. The corresponding sampled system (being T the sampling period) is completely reachable if and only if for each couple λ_i, λ_j of eigenvalues of $\mathbf{A}$ having the same real part, it is:

$$\mathrm{Im}(\lambda_i - \lambda_j) \neq \frac{2k\pi}{T} \qquad k = \pm 1,\ \pm 2,\ \ldots$$

18. For the discrete time-invariant linear system $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,u(k)$, the input sequence $\mathbf{u}(0), \ldots, \mathbf{u}(\bar{k}-1)$ which moves the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(\bar{k})$:

⊗ exists if the system is reachable;
○ exists if $\mathbf{x}(\bar{k}) - e^{\mathbf{A}\bar{k}}\mathbf{x}(0) \in \mathcal{X}^+(\bar{k})$;
○ exists if $\mathbf{x}(\bar{k}) \in \mathcal{X}^+(\bar{k})$;
⊗ exists if $\mathbf{x}(\bar{k}) - \mathbf{A}^{\bar{k}}\mathbf{x}(0) \in \mathcal{X}^+(\bar{k})$;

19. Given the following continuous-time linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \left[\begin{array}{cc|c} -1 & 2 & \mathbf{0} \\ \hline \mathbf{0} & 1 & \mathbf{0} \\ \hline \mathbf{0} & -1 & 1 \end{array}\right] \mathbf{x}(t) + \left[\begin{array}{c} 2 \\ \hline \mathbf{0} \\ \hline \mathbf{0} \end{array}\right] u(t) \\ y(t) = \left[\begin{array}{cc|c} 1 & 1 & \mathbf{0} \end{array}\right] \mathbf{x}(t) \end{cases}$$

Thinking to the block structure of the systems in standard form, it is possible to state that:

⊗ the system is in the reachability standard form;
⊗ the system is in the standard observability form;
○ it is possibile to build a state observer for this system;
○ it is possibile to stabilize the system with a static state feedback;

Using the structural properties of the system compute the transfer function $G(s) = \frac{Y(s)}{U(s)}$ which links the input $U(s) = \mathcal{L}[u(t)]$ to the output $Y(s) = \mathcal{L}[y(t)]$

$$G(s) = \mathbf{c}_1(s\,\mathbf{I} - \mathbf{a}_{11})^{-1}\mathbf{b}_1 = \frac{2}{s+1} \qquad \text{where} \qquad \mathbf{c}_1 = 1,\ \mathbf{a}_{11} = -1,\ \mathbf{b}_1 = 2.$$

20. The Ackermann formula for computing the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of the feedback system can be used

○ for whatever system;
⊗ only if the system is reachable;
○ only if the system is observable;
⊗ only for systems with one input;

21. Given the discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, write the structure of:
a) a *full order closed loop* state estimator:

$$\hat{\mathbf{x}}(k+1) = (\mathbf{A} + \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{L}\mathbf{y}(k)$$

b) a *reducedl order closed loop* state estimator:

$$\hat{\mathbf{x}}(k) = \mathbf{T}\left[\begin{array}{c} \hat{\mathbf{v}}(k) - \mathbf{L}\mathbf{y}(k) \\ \mathbf{y}(k) \end{array}\right]$$

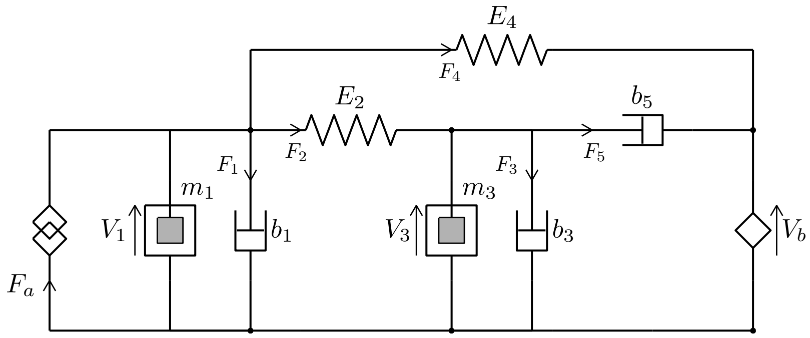
$$\hat{\mathbf{v}}(k+1) = (\overline{\mathbf{A}}_{11} + \mathbf{L}\overline{\mathbf{A}}_{21})\hat{\mathbf{v}}(k) + (\overline{\mathbf{A}}_{12} + \mathbf{L}\overline{\mathbf{A}}_{22} - \overline{\mathbf{A}}_{11}\mathbf{L} - \mathbf{L}\overline{\mathbf{A}}_{21}\mathbf{L})\mathbf{y}(k) + (\mathbf{B}_1 + \mathbf{L}\mathbf{B}_2)\mathbf{u}(k)$$

22. Write the direct Lyapunov stability criterion for continuous-time systems.
Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.
1) If in a neighborhood W *of* $\mathbf{x}_0$ *<u>it exists</u> a function* $V(\mathbf{x}) : W \to \mathcal{R}$ positive definite *with continuous first time-derivatives and if* $\dot{V}(\mathbf{x})$ *is* negative semidefinite, *then the point* $\mathbf{x}_0$ *is* stable *for the nonlinear system* .
2) Moreover, if $\dot{V}(\mathbf{x})$ *is* negative definite, *then the origin is* asymptotically *stable.*

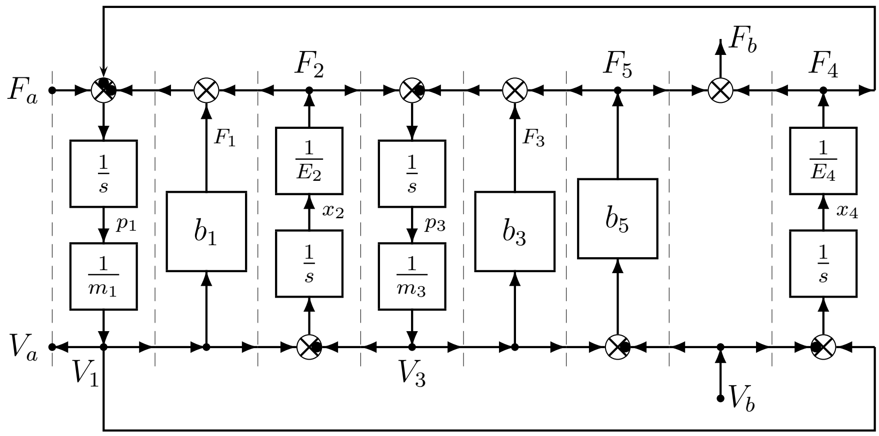
23. Which of the following functions $V(x_1, x_2)$ are positive definite in the vicinity of the origin?

○ $V(x_1, x_2) = x_1^2 x_2^2$;
⊗ $V(x_1, x_2) = x_1^2 + x_2^2$;
○ $V(x_1, x_2) = x_1^2 x_2^2 (x_1^2 + x_2^2)$;
⊗ $V(x_1, x_2) = x_1^2 + x_2^2 - x_1^4 - x_2^4$;

24. Consider the following mechanical system composed by masses m_1, m_3, elasticities E_2, E_4 and dampers b_1, b_2 and b_5. Two inputs act on the system: the force F_a and the velocity V_b. The outputs of the system are: velocity $V_a = V_1$ and the force $F_b = F_4 + F_5$.



The POG model of the given mechanical system has the following structure:



Let $\mathbf{x} = \begin{bmatrix} V_1 & F_2 & V_3 & F_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} F_a & V_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} V_a & F_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & E_4 \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{F}_2 \\ \dot{V}_3 \\ \dot{F}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -b_1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -b_3-b_5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} V_1 \\ F_2 \\ V_3 \\ F_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & b_5 \\ 0 & -1 \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} F_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ F_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & b_5 & 1 \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -b_5 \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} F_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

25. Given an electric element characterized by an output voltage V, an input current I and a nonlinear capacity $C(V)$ which is function of the voltage V. Write : 1) the constitutive equation of the electric element; 2) the differential equation that describes the dynamics of the considered physical element:

1) $Q = C(V)\,V$ 2) $\dfrac{d}{dt}\,[C(V)\,V] = I$

26. Given the following autonomous discrete-time nonlinear system:

$$\begin{cases} x_1(k+1) &= 0.5\sin x_2(k) + x_1(k) \\ x_2(k+1) &= 1 - x_1(k) + x_2^2(k)\sin x_2(k) \end{cases}$$

a) Verify if the point $\mathbf{x}_0 = (1,0)$ is an equilibrium point for the system:

$\mathbf{x}_0 = (1,0)$ is an equilibrium point for the system because the following relations hold:

$$\begin{cases} x_1(k) &= 0.5\sin x_2(k) + x_1(k) \\ x_2(k) &= 1 - x_1(k) + x_2^2(k)\sin x_2(k) \end{cases} \quad \rightarrow \quad \begin{cases} 1 &= 1 \\ 0 &= 1-1 \end{cases}$$

b) Linearize the system in the neighborhood of point $\mathbf{x}_0 = (1,0)$ computing the matrix $\mathbf{A}$ of the corresponding linearized system:

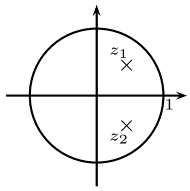
The matrix $\mathbf{A}$ of the linearized system has the following structure:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5\cos x_2 \\ -1 & x_2^2\cos x_2 + 2x_2\sin x_2 \end{bmatrix}_{(x_1=1,\ x_2=0)} = \begin{bmatrix} 1 & 0.5 \\ -1 & 0 \end{bmatrix}$$

c) Study the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = (1,0)$ using the reduced Lyapunov criterion:

The eigenvalues of matrix $\mathbf{A}$ are the roots of following characteristic polynomial:

$$\det[z\mathbf{I}_2 - \mathbf{A}] = 0 \qquad \rightarrow \qquad z^2 - z + 0.5 = 0$$



The roots of the characteristic polynomial are:

$$z_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1-2}\right) = \frac{1}{2}(1 \pm j) = \frac{\sqrt{2}}{2}\, e^{\pm j \arctan\sqrt{7}}$$

Using the reduced Lyapunov criterion it is possible to state that the nonlinear system is stable in the neighborhood of point $\mathbf{x}_0 = (1,0)$ because both the eigenvalues of the system are within the unit circle.

27. Given the following autonomous discrete-time nonlinear system:

$$\begin{cases} x_1(k+1) &= x_1(k) + 3\,x_2^2(k) \\ x_2(k+1) &= 3\,x_2^2(k) \end{cases}$$

It is easy to verify that $\mathbf{x}_0 = (0,0)$ is an equilibrium point per the system. Study the stability of point $\mathbf{x}_0 = (0,0)$ using the "direct" Lyapunov criterion and the function: $V(\mathbf{x}) = x_1^2 + x_2^2$. Eventually, use the La Salle - Krasowskii criterion.

In the neighborhood of point $\mathbf{x}_0 = (0,0)$ the function $V(\mathbf{x}) = x_1^2 + x_2^2$ is surely positive definite. The function $\Delta V(\mathbf{x})$ computed along the system's trajectories is the following:

$$\begin{aligned}\Delta V(\mathbf{x}) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) = (x_1 + 3x_2^2)^2 + 9x_2^4 - x_1^2 - x_2^2 \\ &= x_2^2(18x_2^2 + 6x_1 - 1) \simeq -x_2^2 \leq 0\end{aligned}$$

The function $\Delta V(\mathbf{x})$ is negative semidefinite and therefore, using the "direct" Lyapunov criterion, it can be stated that in the neighborhood of point $\mathbf{x}_0 = (0,0)$ the nonlinear system is stable. The set $\mathcal{N}$ of the points that nullify the function $\Delta V(\mathbf{x})$ is $\mathcal{N} = \{(x_1,\ 0),\ x_1 \in R\}$. For $(x_1,\ x_2) \in \mathcal{N}$ the given system simplifies as follows:

$$\begin{cases} x_1(k+1) = x_1(k) \\ 0 = 0 \end{cases} \qquad \Rightarrow \qquad (x_1,\ x_2) = (x_{10},\ \ 0)$$

The set $\mathcal{N}$ contains the perturbed trajectory $\mathbf{x}(k) = (x_{10},\ \ 0),\ x_{10} \in R$, and therefore the nonlinear system is simply stable in the neighborhood of point $\mathbf{x}_0 = (0,0)$.