System and Control Theory
Test of January 9, 2014
Questions and Exercises

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1. Write the general solution of the linear time-variant differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$, being $\mathbf{x}(t_0)$ the state at time t_0:

$$\mathbf{x}(t) =$$

where matrix $\mathbf{\Phi}(t, t_0)$ is solution of the following matrix differential equation:

$$\frac{d}{dt}\mathbf{\Phi}(t, t_0) = \dots \qquad , \qquad \mathbf{\Phi}(t_0, t_0) = \dots$$

2. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \quad \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \quad \end{bmatrix}$$

The system is: ○ reachable ○ not-reachable ○ observable ○ not-observable

Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \text{Im}\,[\mathcal{B}_R] = \text{Im}\begin{bmatrix} \quad \end{bmatrix}, \qquad \mathcal{E}^- = \text{Im}\,[\mathcal{B}_O] = \text{Im}\begin{bmatrix} \quad \end{bmatrix}.$$

3. The following symbolic representation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \qquad \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

is used for describing a system with the following characteristics:

○ a static system;	○ a continuous-time system;
○ a linear system;	○ a time-invariant system;
○ an autonomous system;	○ a lumped system;

4. Apply the $\mathcal{Z}$-transform to the following *state* function:

$$\mathcal{Z}\left[\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)\right]$$

4.a) write the expression of the transform $\mathbf{x}(z)$ of the state vector $\mathbf{x}(k)$ as a function of the initial state $\mathbf{x}_0$ and the transform $\mathbf{u}(z)$ of the input signal $u(k)$:

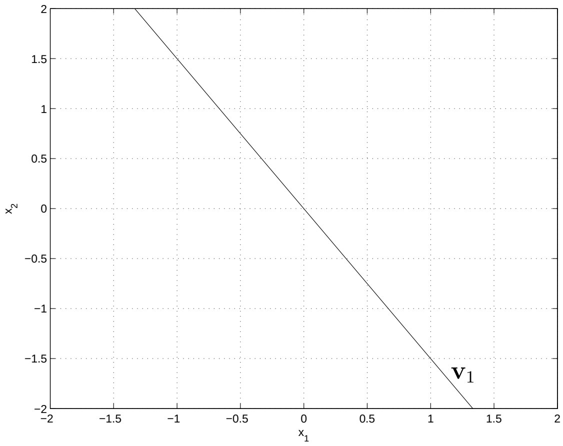
$$\mathbf{x}(z) =$$

4.b) write the expression $\mathbf{x}(k)$ of the discrete function obtained applying the inverse $\mathcal{Z}$ transform to the function $\mathbf{x}$(z):

$$\mathbf{x}(k) = \mathcal{Z}^{-1}[\mathbf{x}(z)] =$$

5. Draw qualitatively the trajectories of a second order dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ characterized by the eigenvalues λ_i and the eigenvectors $\mathbf{v}_i$ shown in the two following boxes.

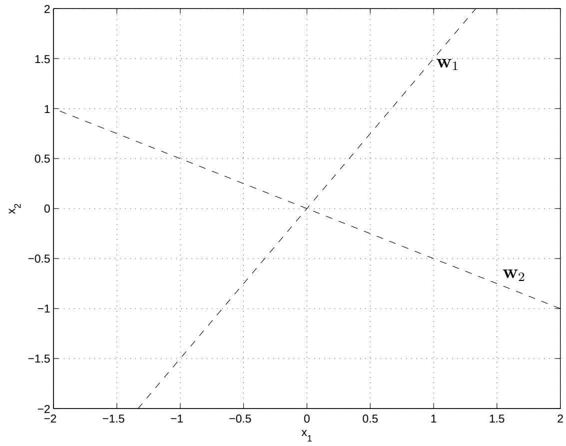
1) Eigenvalue $\lambda = -1$ with multiplicity $r = 2$. The corresponding real eigenvector $\mathbf{v}_1$ is shown in figure.



◯ Node? ◯ Degen. Node? ◯ Focus?

◯ Saddle? ◯ Stable? ◯ Unstable?

2) Eigenvalues: $\lambda_{1,2} = \pm 3j$. Eigenvectors: $\mathbf{v}_1 = \mathbf{w}_1 + \mathbf{w}_2 j$ and $\mathbf{v}_2 = \mathbf{v}_1^*$. The real vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are shown in figure.



◯ Node? ◯ Degen. Node? ◯ Focus?

◯ Saddle? ◯ Stable? ◯ Unstable?

6. Compute, as function of the initial condition $\mathbf{x}_0 = [x_{10},\ x_{20},\ x_{30},\ x_{40}]^{\mathrm{T}}$, the free evolution of the following discrete-time autonomous system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathbf{x}(k) \qquad \mathbf{x}(k) = \begin{bmatrix} \quad & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix}$$

7. Draw the block scheme of the following continuous-time system in the observability canonical form where $\mathbf{x}_o = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$.

$$\begin{cases} \dot{\mathbf{x}}_o(t) = \begin{bmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_o(t) + d_0\, u(t) \end{cases}$$

8. Given the following dynamic system $\mathbf{x}(k{+}1) = \mathbf{A}\,\mathbf{x}(k){+}\mathbf{b}\,u(k)$, $\mathbf{y}(k) = \mathbf{c}\,\mathbf{x}(k){+}d\,u(k)$, compute the function $G(z) = \frac{Y(z)}{U(z)}$ which links the input $U(z) = \mathcal{Z}[u(k)]$ to the output $Y(z) = \mathcal{Z}[y(k)]$:

$$\begin{cases} \mathbf{x}(k{+}1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -4 & -3 & -1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 5 & 0 & 2 & 6 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \end{bmatrix} u(k) \end{cases} \qquad G(z) =$$

Without making specific calculations it can be stated that surely:

- ◯ the system is stable;
- ◯ the system is observable;
- ◯ the system is reachable;
- ◯ the system is stabilizable using a static state feedback;

9. Given a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{u}$, time-invariant, reachable and with only one input. Let $\Delta_{\mathbf{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$ be the characteristic polynomial of matrix $\mathbf{A}$ and let $p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0$ be a monic polynomial freely chosen. Write the expression of vector $\mathbf{k}^{\mathrm{T}}$ which, using the static feedback $\mathbf{u} = \mathbf{k}^{\mathrm{T}}\mathbf{x}$, is able to match the eigenvalues of matrix $\mathbf{A} + \mathbf{b}\mathbf{k}^{\mathrm{T}}$ with the roots of polynomial $p(\lambda)$:

$\mathbf{k}^{\mathrm{T}} =$

where $\mathbf{k}_c^{\mathrm{T}} = [\qquad]$.

10. Write the Ackermann formula for computing the gain vector $\mathbf{l}$ of an asymptotic state observer which freely places the eigenvalues of matrix $\mathbf{A} + \mathbf{l}\mathbf{c}$:

$\mathbf{l} =$

Write the structure of the desired polynomial $p(\lambda)$ and matrix $p(\mathbf{A})$ when three eigenvalues are located in $\lambda = -2$ and other two eigenvalues are located in $\lambda = -5$:

$p(\lambda) =$ $\qquad$ $p(\mathbf{A}) =$

11. Given the following nonlinear differential equation:

$$\dddot{y}(t) + 2\,y(t)\sin\ddot{y}(t) + 3\,\ddot{y}(t)[\dot{y}(t)]^3 + 7\,\cos y(t) = u(t).$$

Chosen $\mathbf{x} = [x_1\;\; x_2\;\; x_3]^{\mathrm{T}} = [y(t)\;\; \dot{y}(t)\;\; \ddot{y}(t)]^{\mathrm{T}}$ as state vector, write the corresponding nonlinear differential equation in the state space:

$$\begin{cases} \dot{x}_1 = \\ \dot{x}_2 = \\ \dot{x}_3 = \end{cases}$$

12. Write the necessary and sufficient condition which guarantees the controllability in k steps of the linear discrete system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$:

13. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$ with period T:

$\mathbf{F} =$ $\qquad\qquad$ $\mathbf{G} =$ $\qquad\qquad$ $\mathbf{H} =$

14. Given the following continuous-time linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

Thinking to the block structure of the systems in standard form it is possible to state that:

- ◯ the system is in the reachability standard form;
- ◯ the system is in the standard observability form;
- ◯ the system is not completely reachable;
- ◯ for this system it is possibile to build a state observer;

Using the structural properties of the system compute the transfer function $G(s) = \frac{Y(s)}{U(s)}$ which links the input $U(s) = \mathcal{L}[u(t)]$ to the output $Y(s) = \mathcal{L}[y(t)]$

$G(s) =$

15. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of:
a) a *full order closed loop* state estimator:

$\dot{\hat{\mathbf{x}}}(t) =$

b) l'andamento of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ starting dall'errore iniziale $\mathbf{e}(0)$:

$\mathbf{e}(t) =$

16. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ which move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the solution $\mathbf{u}$ which minimizes the Euclidean norm:

$\mathbf{u} =$

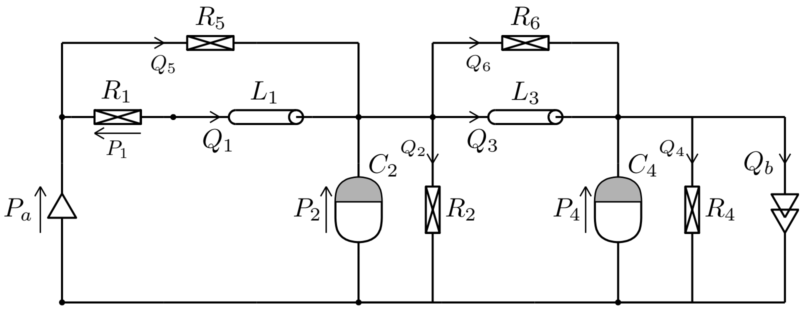
17. Write the instability Lyapunov criterion for continuous-time systems.
Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\ \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *an equilibrium point corresponding to the constant input* $\mathbf{u}_0$*. If*
1) in a neighborhood ...

2) the point $\mathbf{x}_0$ *...*

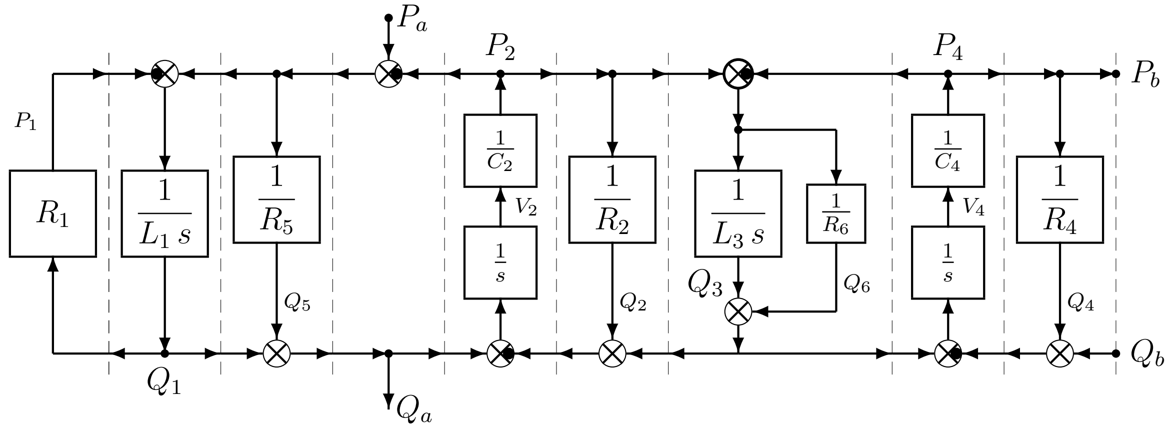
3) ...

then $\mathbf{x}_0$ *is an unstable equilibrium point.*

18. Consider the following hydraulic circuit composed by the hydraulic inductances L_1, L_3, the hydraulic capacities C_2, C_4 and the hydraulic resistances R_1, R_2, R_4, R_5 and R_6. Two inputs act on the system: the pressione P_a and the volume flow rate Q_b. The outputs of the system are: the volume flow rate $Q_a = Q_1 + Q_5$ and the pressione $P_b = P_4$.



The POG model of the given hydraulic circuit has the following structure:



Let $\mathbf{x} = \begin{bmatrix} Q_1 & P_2 & Q_3 & P_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} P_a & Q_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} Q_a & P_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{Q}_1 \\ \dot{P}_2 \\ \dot{Q}_3 \\ \dot{P}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} Q_1 \\ P_2 \\ Q_3 \\ P_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} Q_a \\ P_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

19. Known the output power variables v_1 and v_2, write the name of the dynamic elements $\mathcal{D}_1$ and $\mathcal{D}_2$ and the corresponding energy variables q_1, q_2 that characterize the energetic domains:

	Electrical	**Mech. Trans.**	**Mech. Rot.**	**Hydraulic**
$\mathcal{D}_1$				
q_1				
v_1	V Voltage	v Velocity	ω Ang. Velocity	P Pressure
$\mathcal{D}_2$				
q_2				
v_2	I Current	F Force	τ Torque	Q Volume flow rate

20. Given the following continuous-time autonomous nonlinear system:

$$\begin{cases} \dot{x}_1 &= \beta x_2 - x_1^3 \\ \dot{x}_2 &= \alpha x_2(x_1^2 + x_2^2 - 1) - \beta x_1 \end{cases}$$

It is easy to verify that the origin $\mathbf{x}_0 = (0,\ 0)$ is an equilibrium point for the system.

a) Compute, as a function of parameters α and β, the Jacobian $\mathbf{A}(\mathbf{x})$ of the nonlinear system:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \left[\quad \quad \quad \quad \right]$$

b) Compute, as a function of α and β, the matrix $\mathbf{A}_0$ of the linearized system at the point $\mathbf{x}_0 = (0, 0)$:

$$\mathbf{A}_0 = \left[\quad \quad \right]$$

c) Study, to the variare of the parameters α and β, the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = (0,\ 0)$ using the reduced Lyapunov criterion:

d) For $\alpha = 0$, study for varying parameter β the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = (0, 0)$ using the direct Lyapunov criterion and the function: $V(\mathbf{x}) = x_1^2 + x_2^2$. Eventually, use the La Salle - Krasowskii criterion.