System and Control Theory
Test of January 9, 2014
Questions and Exercises

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1. Write the general solution of the linear time-variant differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$, being $\mathbf{x}(t_0)$ the state at time t_0:

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau$$

where matrix $\mathbf{\Phi}(t, t_0)$ is solution of the following matrix differential equation:

$$\frac{d}{dt}\mathbf{\Phi}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0), \qquad \mathbf{\Phi}(t_0, t_0) = \mathbf{I}$$

2. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \mathbf{1} & \mathbf{0} & 1 & \mathbf{1} & 1 & 1 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{1} & 0 & 1 \\ \mathbf{0} & \mathbf{1} & 0 & \mathbf{0} & 0 & 1 \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

The system is: $\otimes$ reachable $\bigcirc$ not-reachable $\bigcirc$ observable $\otimes$ not-observable

Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \mathrm{Im}\mathcal{R}^+ = \mathrm{Im}\,[\mathcal{B}_R] = \mathrm{Im}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathcal{E}^- = \ker\mathcal{O}^- = \mathrm{Im}\,[\mathcal{B}_O] = \mathrm{Im}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. The following symbolic representation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \qquad \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$$

is used for describing a system with the following characteristics:

$\bigcirc$ a static system; $\otimes$ a continuous-time system;
$\otimes$ a linear system; $\bigcirc$ a time-invariant system;
$\otimes$ an autonomous system; $\otimes$ a lumped system;

4. Apply the $\mathcal{Z}$-transform to the following *state* function:

$$\mathcal{Z}\,[\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)]$$

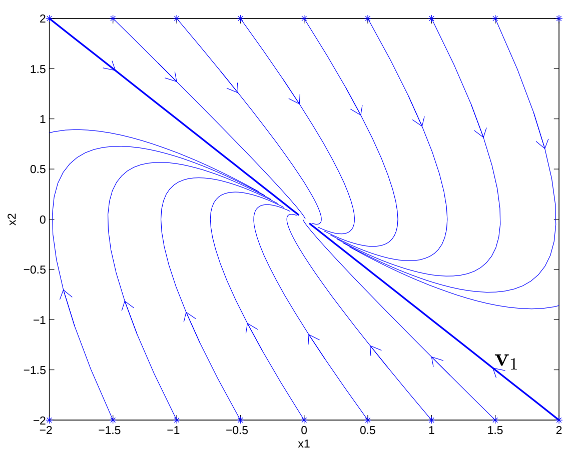
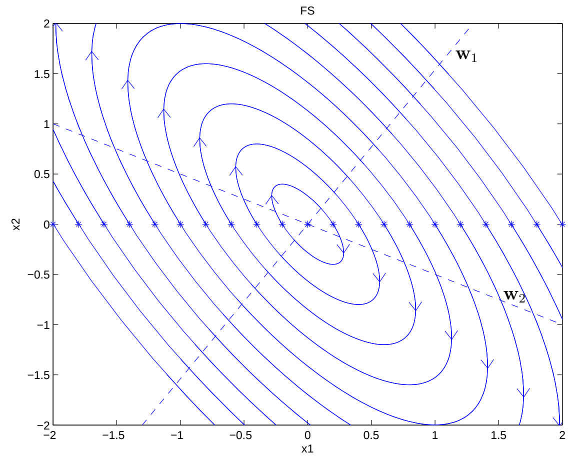
4.a) write the expression of the transform $\mathbf{x}(z)$ of the state vector $\mathbf{x}(k)$ as a function of the initial state $\mathbf{x}_0$ and the transform $\mathbf{u}(z)$ of the input signal $u(k)$:

$$\mathbf{x}(z) = (z\mathbf{I} - \mathbf{A})^{-1} z\,\mathbf{x}_0 + (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\,\mathbf{u}(z)$$

4.b) write the expression $\mathbf{x}(k)$ of the discrete function obtained applying the inverse $\mathcal{Z}$ transform to the function $\mathbf{x}$(z):

$$\mathbf{x}(k) = \mathcal{Z}^{-1}[\mathbf{x}(z)] = \mathbf{A}^k\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{k-j-1}\mathbf{B}\mathbf{u}(j)$$

5. Draw qualitatively the trajectories of a second order dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ characterized by the eigenvalues λ_i and the eigenvectors $\mathbf{v}_i$ shown in the two following boxes.

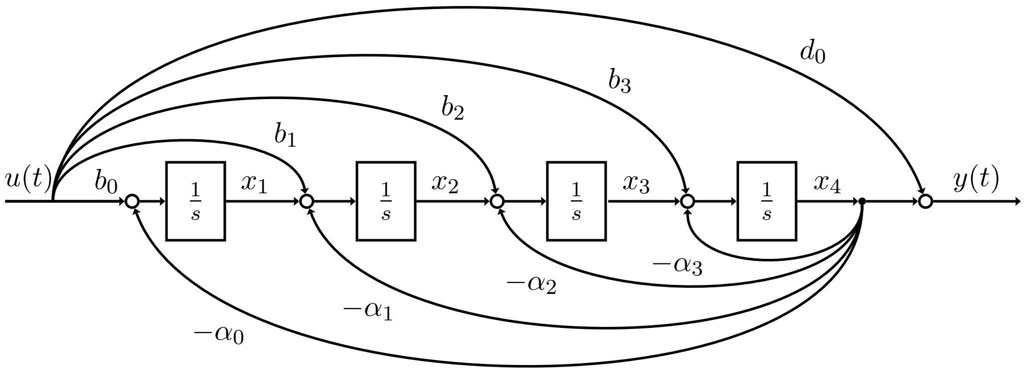
1) Eigenvalue $\lambda = -1$ with multiplicity $r = 2$. The corresponding real eigenvector $\mathbf{v}_1$ is shown in figure.	2) Eigenvalues: $\lambda_{1,2} = \pm 3j$. Eigenvectors: $\mathbf{v}_1 = \mathbf{w}_1 + \mathbf{w}_2 j$ and $\mathbf{v}_2 = \mathbf{v}_1^*$. The real vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are shown in figure.
	
◯ Node? ⊗ Degen. Node? ◯ Focus?	◯ Node? ◯ Degen. Node? ⊗ Focus?
◯ Saddle? ⊗ Stable? ◯ Unstable?	◯ Saddle? ⊗ Stable? ◯ Unstable?

6. Compute, as function of the initial condition $\mathbf{x}_0 = [x_{10},\ x_{20},\ x_{30},\ x_{40}]^{\mathrm{T}}$, the free evolution of the following discrete-time autonomous system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathbf{x}(k) \qquad \mathbf{x}(k) = \begin{bmatrix} 2^k & k\,2^{k-1} & \frac{k(k-1)}{2}\,2^{k-2} & \frac{k(k-1)(k-3)}{6}\,2^{k-3} \\ 0 & 2^k & k\,2^{k-1} & \frac{k(k-1)}{2}\,2^{k-2} \\ 0 & 0 & 2^k & k\,2^{k-1} \\ 0 & 0 & 0 & 2^k \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix}$$

7. Draw the block scheme of the following continuous-time system in the observability canonical form where $\mathbf{x}_o = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$.

$$\begin{cases} \dot{\mathbf{x}}_o(t) = \begin{bmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_o(t) + d_0\, u(t) \end{cases}$$



8. Given the following dynamic system $\mathbf{x}(k{+}1) = \mathbf{A}\,\mathbf{x}(k){+}\mathbf{b}\,u(k)$, $\mathbf{y}(k) = \mathbf{c}\,\mathbf{x}(k){+}d\,u(k)$, compute the function $G(z) = \frac{Y(z)}{U(z)}$ which links the input $U(z) = \mathcal{Z}[u(k)]$ to the output $Y(z) = \mathcal{Z}[y(k)]$:

$$\left\{\begin{array}{l} \mathbf{x}(k{+}1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -4 & -3 & -1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k) \\ \\ y(k) = \begin{bmatrix} 5 & 0 & 2 & 6 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \end{bmatrix} u(k) \end{array}\right. \qquad \begin{aligned} G(z) &= \frac{6z^3 + 2z^2 + 5}{z^4 + z^3 + 3z^2 + 4z + 2} + 1 \\ &= \frac{z^4 + 7z^3 + 5z^2 + 4z^2 + 7}{z^4 + z^3 + 3z^2 + 4z + 2} \end{aligned}$$

Without making specific calculations it can be stated that surely:

- ○ the system is stable;
- ○ the system is observable;
- ⊗ the system is reachable;
- ⊗ the system is stabilizable using a static state feedback;

9. Given a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{u}$, time-invariant, reachable and with only one input. Let $\Delta_{\mathbf{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$ be the characteristic polynomial of matrix $\mathbf{A}$ and let $p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0$ be a monic polynomial freely chosen. Write the expression of vector $\mathbf{k}^{\mathrm{T}}$ which, using the static feedback $\mathbf{u} = \mathbf{k}^{\mathrm{T}}\mathbf{x}$, is able to match the eigenvalues of matrix $\mathbf{A} + \mathbf{b}\mathbf{k}^{\mathrm{T}}$ with the roots of polynomial $p(\lambda)$:

$$\mathbf{k}^{\mathrm{T}} = \mathbf{k}_c^{\mathrm{T}} \left\{ \begin{bmatrix} \mathbf{b}, & \mathbf{A}\mathbf{b}, & \ldots, & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & 1 \\ \alpha_2 & \ldots & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \right\}^{-1}$$

where $\mathbf{k}_c^{\mathrm{T}} = \begin{bmatrix} \alpha_0 - d_0, & \alpha_1 - d_1, & \ldots, & \alpha_{n-1} - d_{n-1} \end{bmatrix}$.

10. Write the Ackermann formula for computing the gain vector $\mathbf{l}$ of an asymptotic state observer which freely places the eigenvalues of matrix $\mathbf{A} + \mathbf{l}\mathbf{c}$:

$$\mathbf{l} = -p(\mathbf{A})(\mathcal{O}^-)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = -p(\mathbf{A})\mathbf{q}$$

Write the structure of the desired polynomial $p(\lambda)$ and matrix $p(\mathbf{A})$ when three eigenvalues are located in $\lambda = -2$ and other two eigenvalues are located in $\lambda = -5$:

$$p(\lambda) = (\lambda + 2)^3(\lambda + 5)^2, \qquad\qquad p(\mathbf{A}) = (\mathbf{A} + 2\mathbf{I})^3(\mathbf{A} + 5\mathbf{I})^2$$

11. Given the following nonlinear differential equation:

$$\dddot{y}(t) + 2\,y(t)\sin\ddot{y}(t) + 3\,\ddot{y}(t)[\dot{y}(t)]^3 + 7\,\cos y(t) = u(t).$$

Chosen $\mathbf{x} = [x_1\;\; x_2\;\; x_3]^{\mathrm{T}} = [y(t)\;\; \dot{y}(t)\;\; \ddot{y}(t)]^{\mathrm{T}}$ as state vector, write the corresponding nonlinear differential equation in the state space:

$$\left\{\begin{array}{lcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& x_3 \\ \dot{x}_3 &=& -2\,x_1\sin x_3 - 3\,x_3\,x_2^3 - 7\,\cos x_1 + u(t) \end{array}\right.$$

12. Write the necessary and sufficient condition which guarantees the controllability in k steps of the linear discrete system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$:

$$\mathrm{Im}\mathbf{A}^k \subseteq \mathrm{Im}[\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{k-1}\mathbf{B}] = \mathcal{X}^+(k)$$

13. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$ with period T:

$$\mathbf{F} = e^{\mathbf{A}T}, \qquad \mathbf{G} = \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\, d\sigma, \qquad \mathbf{H} = \mathbf{C}$$

14. Given the following continuous-time linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \left[\begin{array}{c|cc} 3 & \mathbf{0} & \mathbf{0} \\ \hline 0 & -1 & 1 \\ 1 & 0 & -2 \end{array}\right] \mathbf{x}(t) + \left[\begin{array}{c} 1 \\ \hline 0 \\ 1 \end{array}\right] u(t) \\ y(t) = \left[\begin{array}{c|cc} 2 & \mathbf{0} & \mathbf{0} \end{array}\right] \mathbf{x}(t) \end{cases}$$

Thinking to the block structure of the systems in standard form it is possible to state that:

- ○ the system is in the reachability standard form;
- ⊗ the system is in the standard observability form;
- ○ the system is not completely reachable;
- ⊗ for this system it is possibile to build a state observer;

Using the structural properties of the system compute the transfer function $G(s) = \frac{Y(s)}{U(s)}$ which links the input $U(s) = \mathcal{L}[u(t)]$ to the output $Y(s) = \mathcal{L}[y(t)]$

$$G(s) = \mathbf{c}_1(s\,\mathbf{I} - \mathbf{a}_{11})^{-1}\mathbf{b}_1 = \frac{2}{s-3} \qquad \text{where} \qquad \mathbf{c}_1 = 2,\ \mathbf{a}_{11} = 3,\ \mathbf{b}_1 = 1.$$

15. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of:

a) a *full order closed loop* state estimator:

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} + \mathbf{LC})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t)$$

b) l'andamento of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ starting dall'errore iniziale $\mathbf{e}(0)$:

$$\mathbf{e}(t) = e^{(\mathbf{A}+\mathbf{LC})t}\,\mathbf{e}(0)$$

16. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ which move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the solution $\mathbf{u}$ which minimizes the Euclidean norm:

$$\mathbf{u} = (\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}[\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0)]$$

17. Write the instability Lyapunov criterion for continuous-time systems.

Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\ \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *an equilibrium point corresponding to the constant input* $\mathbf{u}_0$. *If*

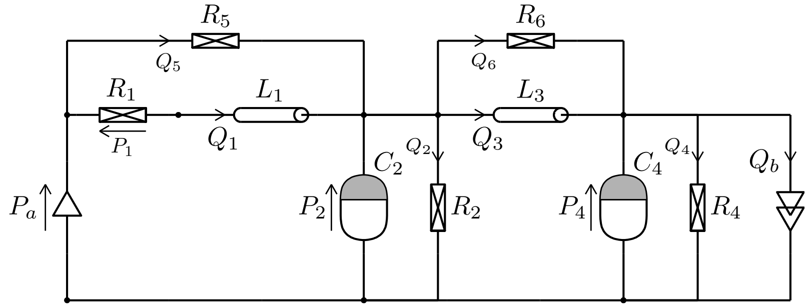
1) in a neighborhood W *of* $\mathbf{x}_0$ *it exists a function* $V(\mathbf{x}) : W \to \mathcal{R}$ *continua with continuous first time-derivatives and zero in* $\mathbf{x}_0$*;*

2) the point $\mathbf{x}_0$ *is an accumulation point for the set of points* $\mathbf{x} \in W$ *in which it is* $V(\mathbf{x}) > 0$*;*

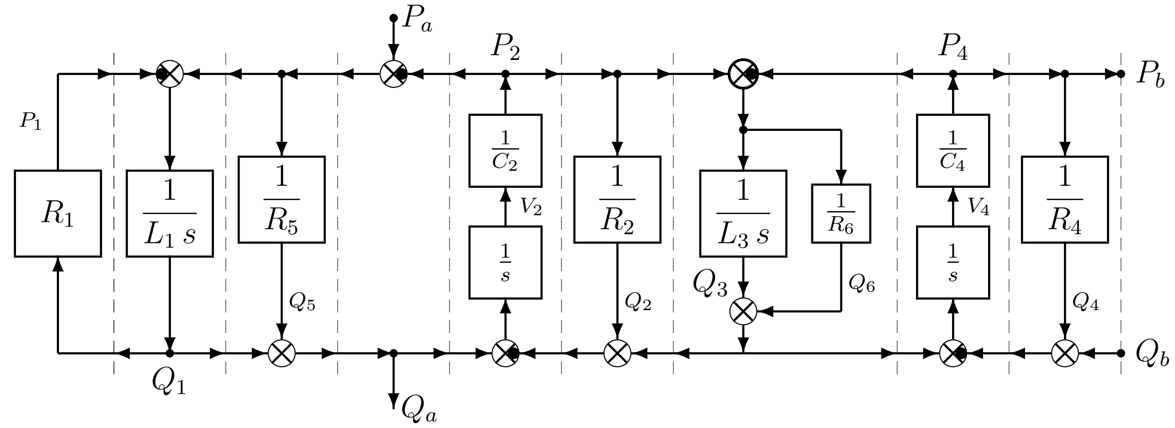
3) $\dot{V}(\mathbf{x})$ *is* positive definite *in* W*;*

then $\mathbf{x}_0$ *is an unstable equilibrium point.*

18. Consider the following hydraulic circuit composed by the hydraulic inductances L_1, L_3, the hydraulic capacities C_2, C_4 and the hydraulic resistances R_1, R_2, R_4, R_5 and R_6. Two inputs act on the system: the pressione P_a and the volume flow rate Q_b. The outputs of the system are: the volume flow rate $Q_a = Q_1 + Q_5$ and the pressione $P_b = P_4$.



The POG model of the given hydraulic circuit has the following structure:



Let $\mathbf{x} = \begin{bmatrix} Q_1 & P_2 & Q_3 & P_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} P_a & Q_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} Q_a & P_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{Q}_1 \\ \dot{P}_2 \\ \dot{Q}_3 \\ \dot{P}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_1 & -1 & 0 & 0 \\ 1 & -\frac{1}{R_2}-\frac{1}{R_5}-\frac{1}{R_6} & -1 & \frac{1}{R_6} \\ 0 & 1 & 0 & -1 \\ 0 & \frac{1}{R_6} & 1 & -\frac{1}{R_4}-\frac{1}{R_6} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} Q_1 \\ P_2 \\ Q_3 \\ P_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{R_5} & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} Q_a \\ P_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & -\frac{1}{R_5} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_5} & 0 \\ 0 & 0 \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} P_a \\ Q_b \end{bmatrix}}_{\mathbf{u}}$$

19. Known the output power variables v_1 and v_2, write the name of the dynamic elements $\mathcal{D}_1$ and $\mathcal{D}_2$ and the corresponding energy variables q_1, q_2 that characterize the energetic domains:

	Electrical		**Mech. Trans.**		**Mech. Rot.**		**Hydraulic**	
$\mathcal{D}_1$	C	Capacity	M	Mass	J	Inertia	C_I	Hydr. Capacity
q_1	Q	Charge	p	Momentum	p	Ang. Momentum	V	Volume
v_1	V	Voltage	v	Velocity	ω	Ang. Velocity	P	Pressure
$\mathcal{D}_2$	L	Inductance	E	Elasticity	E	Tor. Elasticity	L_I	Hydr. Inductance
q_2	ϕ	Flux	x	Displacement	θ	Ang. Displacement	ϕ_I	Hudr. Flux
v_2	I	Current	F	Force	τ	Torque	Q	Volume flow rate

20. Given the following continuous-time autonomous nonlinear system:

$$\begin{cases} \dot{x}_1 &= \beta x_2 - x_1^3 \\ \dot{x}_2 &= \alpha x_2(x_1^2 + x_2^2 - 1) - \beta x_1 \end{cases}$$

It is easy to verify that the origin $\mathbf{x}_0 = (0,\ 0)$ is an equilibrium point for the system.

a) Compute, as a function of parameters α and β, the Jacobian $\mathbf{A}(\mathbf{x})$ of the nonlinear system:

The Jacobian $\mathbf{A}(\mathbf{x})$ has the following structure:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} -3x_1^2 & \beta \\ 2\alpha x_1 x_2 - \beta & \alpha(x_1^2 + 3x_2^2 - 1) \end{bmatrix}$$

b) Compute, as a function of α and β, the matrix $\mathbf{A}_0$ of the linearized system at the point $\mathbf{x}_0 = (0, 0)$:

The matrix $\mathbf{A}_0$ of the linearized system has the following structure:

$$\mathbf{A}_0 = \begin{bmatrix} 0 & \beta \\ -\beta & -\alpha \end{bmatrix}$$

c) Study, to the variare of the parameters α and β, the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = (0,\ 0)$ using the reduced Lyapunov criterion:

The characteristic polynomial of matrix $\mathbf{A}_0$ is:

$$\Delta_{\mathbf{A}_0}(s) = s^2 + \alpha\, s + \beta^2$$

Using the reduced Lyapunov criterion it can be stated that: 1) for $\alpha > 0$ and $\beta \neq 0$ the equilibrium point $\mathbf{x}_0 = (0, 0)$ of the nonlinear system is asymptotically stable; 2) for $\alpha < 0$ and $\beta \neq 0$ the equilibrium point $\mathbf{x}_0$ is unstable; 3) for $\alpha = 0$ and $\beta \neq 0$ the criterion cannot be used; 4) for $\beta = 0$ the criterion cannot be used.

d) For $\alpha = 0$, study for varying parameter β the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_0 = (0, 0)$ using the direct Lyapunov criterion and the function: $V(\mathbf{x}) = x_1^2 + x_2^2$. Eventually, use the La Salle - Krasowskii criterion.

In the neighborhood of point $\mathbf{x}_0 = (0, 0)$ the function $V(\mathbf{x}) = x_1^2 + x_2^2$ is surely positive definite. Set $\alpha = 0$, the function $\dot{V}(\mathbf{x})$ computed along the system's trajectories is the following:

$$\begin{aligned} \dot{V}(\mathbf{x}) &= 2x_1(\beta x_2 - x_1^3) + 2x_2(-\beta x_1) \\ &= 2\beta x_1 x_2 - 2x_1^4 - 2\beta x_1 x_2 = -2x_1^4 \leq 0 \end{aligned}$$

The function $\dot{V}(\mathbf{x})$ is negative semidefinite and therefore, using the "direct" Lyapunov criterion, it can be stated that in the neighborhood of point $\mathbf{x}_0 = (0, 0)$ the nonlinear system is stable for all the values of β. The set $\mathcal{N}$ of all the points that nullify the function $\dot{V}(\mathbf{x})$ is $\mathcal{N} = \{(0,\ x_2),\ x_2 \in R\}$. For $(x_1,\ x_2) \in \mathcal{N}$ and $\alpha = 0$ the given system simplifies as follows:

$$\begin{cases} 0 = \beta x_2 \\ \dot{x}_2 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} (x_1,\ x_2) = (0,\ 0) & \text{if} \quad \beta = 0 \\ (x_1,\ x_2) = (0,\ x_2) & \text{if} \quad \beta \neq 0 \end{cases}$$

For $\beta \neq 0$ the set $\mathcal{N}$ does not contain perturbed trajectories and therefore the nonlinear system is asymptotically stable in the neighborhood of point $\mathbf{x}_0 = (0, 0)$. For $\beta = 0$ the set $\mathcal{N}$ contains the perturbed trajectory $(0,\ x_2)$ and therefore the nonlinear system is simply stable in the neighborhood of point $\mathbf{x}_0 = (0, 0)$.