

System and Control Theory
Test of January 23, 2013
Questions and Exercises

Name:	
Nr. Mat.	
Signature:	

1. Write the discrete time behavior of the output function $\mathbf{y}(k)$, solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ and the static equation $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$ starting from the initial condition $\mathbf{x}(h)$ at time h :

$$\mathbf{y}(k) =$$

2. Write the general solution of the following time-varying differential equation $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$, being $\mathbf{x}(t_0)$ the state at time t_0 :

$$\mathbf{x}(t) =$$

where $\Phi(t, \tau)$ is the state transition matrix of the system in the time interval $[\tau, t]$.

3. Compute the reachability matrix \mathcal{R}^+ and the observability matrix \mathcal{O}^- of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) \end{cases} \quad \mathcal{R}^+ = \begin{bmatrix} \\ \\ \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \\ \\ \end{bmatrix}$$

The system is: reachable? not-reachable? observable? not-observable?

Provide a base of the reachable subspace \mathcal{X}^+ and a base of the not-observable subspace \mathcal{E}^- :

$$\mathcal{X}^+ = \text{Im} \begin{bmatrix} \\ \\ \end{bmatrix}, \quad \mathcal{E}^- = \text{Im} \begin{bmatrix} \\ \\ \end{bmatrix}.$$

4. Write the definition of the characteristic polynomial $\Delta_{\mathbf{A}}(\lambda)$ of a matrix $\mathbf{A} \in \mathbf{R}^{n \times n}$:

$$\Delta_{\mathbf{A}}(\lambda) =$$

The roots of the characteristic polynomial:

- | | |
|--|---|
| <input type="radio"/> are the eigenvalues of matrix \mathbf{A} ; | <input type="radio"/> always have multiplicity equal to 1; |
| <input type="radio"/> are eigenvectors of matrix \mathbf{A} ; | <input type="radio"/> always are in a number equal to n ; |
| <input type="radio"/> can be complex conjugate; | <input type="radio"/> always are real; |

5. Given a continuous-time linear system: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$. Apply the Laplace transform to the system and provide the expression of the transform $\mathbf{y}(s)$ of the output vector $\mathbf{y}(t)$ corresponding to the *forced evolution* of the system:

$$\mathbf{y}(s) =$$

6. Given an autonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ of the fourth order where the matrix \mathbf{A} can be given the “real” Jordan form $\mathbf{A}_R = \mathbf{T}_R^{-1}\mathbf{A}\mathbf{T}_R$ using the state space transformation $\mathbf{x} = \mathbf{T}_R\bar{\mathbf{x}}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0.4 & -2.2 & 0.4 & 1.6 \\ -4 & -2 & -3 & -4 \\ 0.8 & 0.6 & 0.8 & -1.8 \end{bmatrix}, \quad \mathbf{A}_R = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{T}_R = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -2 \\ 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- a) Write the eigenvalues λ_i and the eigenvectors \mathbf{v}_i that characterize the matrix \mathbf{A} :

$$\begin{aligned} \lambda_1 = & \\ \lambda_2 = & \\ \lambda_3 = & \\ \lambda_4 = & \end{aligned} \quad \mathbf{v}_1 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

<p>b.1) Draw qualitatively the trajectories generated in the plane \mathcal{X}_a by the two complex conjugate eigenvalues of the dynamic system:</p> <div style="border: 1px solid black; width: 100%; height: 100%; position: relative;"> <div style="position: absolute; bottom: 5px; left: 5px;">\mathcal{X}_a</div> </div>	<p>b.2) Draw qualitatively the trajectories generated in the plane \mathcal{X}_b by the two real eigenvalues of the dynamic system:</p> <div style="border: 1px solid black; width: 100%; height: 100%; position: relative;"> <div style="position: absolute; bottom: 5px; left: 5px;">\mathcal{X}_b</div> </div>
<p>c.1) Write the type of trajectories:</p> <p><input type="radio"/> Node? <input type="radio"/> Focus? <input type="radio"/> Saddle?</p> <p><input type="radio"/> Stable? <input type="radio"/> Unstable?</p>	<p>c.2) Write the type of trajectories:</p> <p><input type="radio"/> Node? <input type="radio"/> Focus? <input type="radio"/> Saddle?</p> <p><input type="radio"/> Stable? <input type="radio"/> Unstable?</p>
<p>d.1) Provide a base \mathcal{B}_a of plane $\mathcal{X}_a = \text{Im}\mathcal{B}_a$:</p> $\mathcal{B}_a = \begin{bmatrix} \\ \\ \end{bmatrix}$	<p>d.2) Provide a base \mathcal{B}_b of plane $\mathcal{X}_b = \text{Im}\mathcal{B}_b$:</p> $\mathcal{B}_b = \begin{bmatrix} \\ \\ \end{bmatrix}$

7. Relatively to a discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ that has at least an eigenvalue in the origin, it can be stated that:

- if the system is completely observable then it is also completely constructable;
- if the system is completely constructable then it is also completely observable;
- if the system is completely controllable then it is also completely reachable;
- if the system is completely reachable then it is also completely controllable;
- the system can be asymptotically stable;

8. An “open loop” state estimator can be used
- if and only if the system is observable;
 - if and only if the system is asymptotically stable;
 - if and only if the unstable part of the system is observable;
 - if and only if the not-observable part of the system is asymptotically stable;
9. A system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is “stabilizable” using a static state feedback
- if the system is stable;
 - if the system is observable;
 - if the unstable part of the system is reachable;
 - if the part not-reachable of the system is stable;
10. The Ackermann formula for computing the vector \mathbf{k}^T allowing the free positioning of the eigenvalues of the feedback system can be used:
- for whatever system;
 - only for systems with one input;
 - only if the system is reachable;
 - also for unstable systems;

Provide the explicit description of the Ackermann formula and the desired polynomial $p(\lambda)$ when all the n eigenvalues of the system must be located in $\lambda = -3$:

$$\mathbf{k}^T = \qquad p(\lambda) =$$

11. Given a SISO linear system of the fourth order ($n = 4$), completely reachable, characterized by matrices \mathbf{A} , \mathbf{b} and \mathbf{c} .
- a) Write the structure of the matrices \mathbf{A}_c , \mathbf{b}_c and \mathbf{c}_c of the corresponding controllability canonical form. Let $p(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ the characteristic polynomial of matrix \mathbf{A} .

$$\mathbf{A}_c = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}, \quad \mathbf{b}_c = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \mathbf{c}_c = \begin{bmatrix} & & & \end{bmatrix}$$

- b) Moreover, write the structure of matrix \mathbf{T} which, together with the space transformation $\mathbf{x} = \mathbf{T}\mathbf{x}_c$, brings the system in controllability canonical form.

$$\mathbf{T} =$$

12. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices \mathbf{F} , \mathbf{G} and \mathbf{H} that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$ with period T :

$$\mathbf{F} = \qquad \mathbf{G} = \qquad \mathbf{H} =$$

13. Write the structure of the matrix \mathbf{P}^{-1} of the state space transformation $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ which brings a not-observable system in the standard observability form:

$$\mathbf{P}^{-1} =$$

Moreover, write the block structure of the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$:

$$\bar{\mathbf{A}} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}, \quad \bar{\mathbf{C}} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Write the simplified form of transfer matrix $\mathbf{H}(s)$ of the system \mathcal{S} as a function of the submatrices $\mathbf{A}_{i,j}$, \mathbf{B}_i and \mathbf{C}_j that characterize the system $\bar{\mathcal{S}} = (\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$:

$$\mathbf{H}(s) =$$

14. Write the “*separation property*” of the regulator:

15. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions \mathbf{u} that move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0, k]$ write the structure of the solution \mathbf{u} which minimizes the Euclidean norm $\|\mathbf{u}\|$:

$$\mathbf{u} =$$

16. Write the *Heymann Lemma*:

17. Consider a continuous-time nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ and let \mathbf{x}_0 an equilibrium point of the system for constant input \mathbf{u}_0 . Write the linear part of the Taylor series expansion of the function $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ in the neighborhood of point $(\mathbf{x}_0, \mathbf{u}_0)$:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) =$$

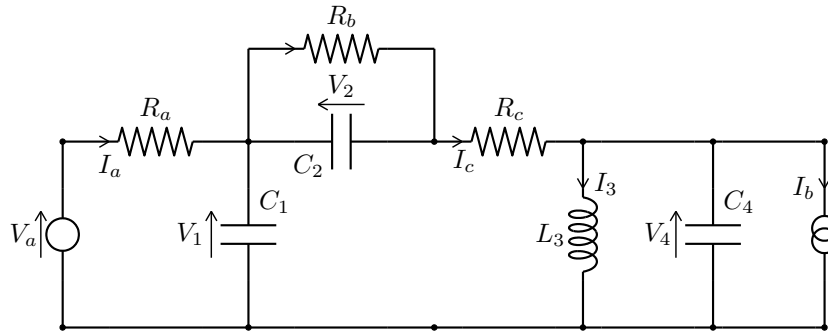
18. Compute the 3 equilibrium points $\bar{\mathbf{x}}_1$, $\bar{\mathbf{x}}_2$ and $\bar{\mathbf{x}}_3$ of the following *continuous-time* nonlinear system:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_1(t)(x_1^2(t) - 1) - 2x_2(t) \end{cases} \Rightarrow \begin{cases} \bar{\mathbf{x}}_1 = (\quad , \quad) \\ \bar{\mathbf{x}}_2 = (\quad , \quad) \\ \bar{\mathbf{x}}_3 = (\quad , \quad) \end{cases}$$

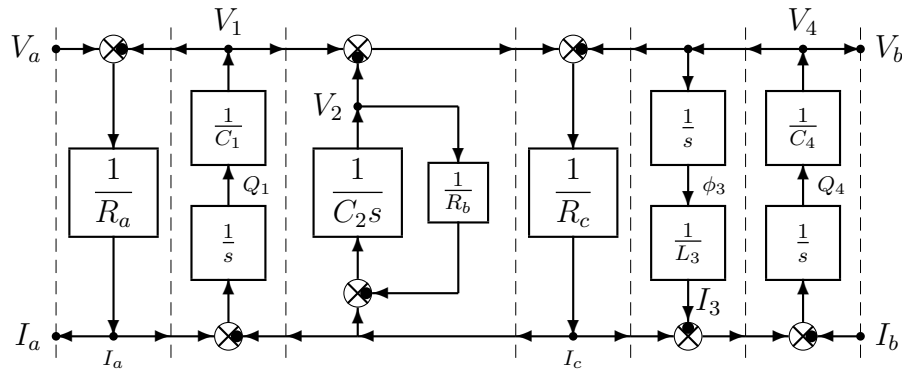
19. Compute the 3 equilibrium points $\tilde{\mathbf{x}}_1$, $\tilde{\mathbf{x}}_2$ and $\tilde{\mathbf{x}}_3$ of the following *discrete-time* nonlinear system:

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = x_1(k)(x_1^2(k) - 1) - 2x_2(k) \end{cases} \Rightarrow \begin{cases} \tilde{\mathbf{x}}_1 = (\quad , \quad) \\ \tilde{\mathbf{x}}_2 = (\quad , \quad) \\ \tilde{\mathbf{x}}_3 = (\quad , \quad) \end{cases}$$

20. Consider the following electric circuit composed by the capacities C_1, C_2, C_4 , the inductance L_3 and the resistances R_a, R_b and R_c . Two inputs act on the system: the voltage V_a and the current I_b . The outputs of the system are: the current I_a and the voltage V_b .



The POG model of the given electric circuit has the following structure:



Let $\mathbf{x} = [V_1 \ V_2 \ I_3 \ V_4]^T$ be the state vector, $\mathbf{u} = [V_a \ I_b]^T$ the input vector and $\mathbf{y} = [I_a \ V_b]^T$ the output vector. Write the corresponding dynamic system $\bar{\mathbf{L}}\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \bar{\mathbf{C}}\mathbf{x} + \bar{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{\bar{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{\bar{\mathbf{A}}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{\bar{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \\ \end{bmatrix}}_{\bar{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \\ \end{bmatrix}}_{\bar{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

21. Write the energy variables q_1, q_2 and the output power variables v_1 and v_2 of the dynamic elements \mathcal{D}_1 and \mathcal{D}_1 that characterize the energetic domains:

	Electrical	Mech. Trans.	Mech. Rot.	Hydraulic
\mathcal{D}_1	C Capacity	M Mass	J Inertia	C_I Hydr. Capacity
q_1				
v_1				
\mathcal{D}_2	L Inductance	E Elasticity	E Tor. Elasticity	L_I Hydr. Inductance
q_2				
v_2				

