System and Control Theory
Test of January 23, 2013
Questions and Exercises

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1. Write the discrete time behavior of the output function $\mathbf{y}(k)$, solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ and the static equation $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$ starting from the initial condition $\mathbf{x}(h)$ at time h:

$$\mathbf{y}(k) = \mathbf{C}\,\mathbf{A}^{k-h}\mathbf{x}(h) + \mathbf{C}\sum_{j=h}^{k-1}\mathbf{A}^{k-j-1}\mathbf{B}\mathbf{u}(j) + \mathbf{D}\mathbf{u}(k)$$

2. Write the general solution of the following time-varying differential equation $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$, being $\mathbf{x}(t_0)$ the state at time t_0:

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t}\mathbf{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\mathbf{d}\tau$$

where $\mathbf{\Phi}(t,\tau)$ is the state transition matrix of the system in the time interval $[\tau,\ t]$.

3. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}u(t) \\ y(t) = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}\mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} 1 & -3 & 1 \\ -1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathcal{O}^- = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The system is: ○ reachable? ⊗ not-reachable? ○ observable? ⊗ not-observable?

Provide a base of the reachable subspace $\mathcal{X}^+$ and a base of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \mathrm{Im}\begin{bmatrix} 1 & -3 \\ -1 & 3 \\ 1 & 1 \end{bmatrix}, \qquad \mathcal{E}^- = \mathrm{Im}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

4. Write the definition of the characteristic polynomial $\Delta_{\mathbf{A}}(\lambda)$ of a matrix $\mathbf{A} \in \mathbf{R}^{n\times n}$:

$$\Delta_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - A)$$

The roots of the characteristic polynomial:

⊗ are the eigenvalues of matrix $\mathbf{A}$;
○ are eigenvectors of matrix $\mathbf{A}$;
⊗ can be complex conjugate;
○ always have multiplicity equal to 1;
⊗ always are in a number equal to n;
○ always are real;

5. Given a continuous-time linear system: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$. Apply the Laplace transform to the system and provide the expression of the transform $\mathbf{y}(s)$ of the output vector $\mathbf{y}(t)$ corresponding to the *forced evolution* of the system:

$$\mathbf{y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{u}(z)$$

6. Given an autonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ of the forth order where the matrix $\mathbf{A}$ can be given the "real" Jordan form $\mathbf{A}_R = \mathbf{T}_R^{-1}\mathbf{A}\mathbf{T}_R$ using the state space transformation $\mathbf{x} = \mathbf{T}_R\bar{\mathbf{x}}$:

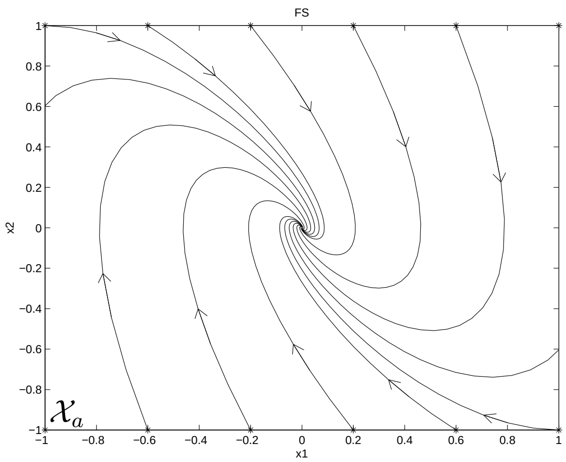
$$\mathbf{A}=\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0.4 & -2.2 & 0.4 & 1.6 \\ -4 & -2 & -3 & -4 \\ 0.8 & 0.6 & 0.8 & -1.8 \end{bmatrix}, \qquad \mathbf{A}_R=\begin{bmatrix} -2 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \qquad \mathbf{T}_R=\begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -2 \\ 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

a) Write the eigenvalues λ_i and the eigenvectors $\mathbf{v}_i$ that characterize the matrix $\mathbf{A}$:

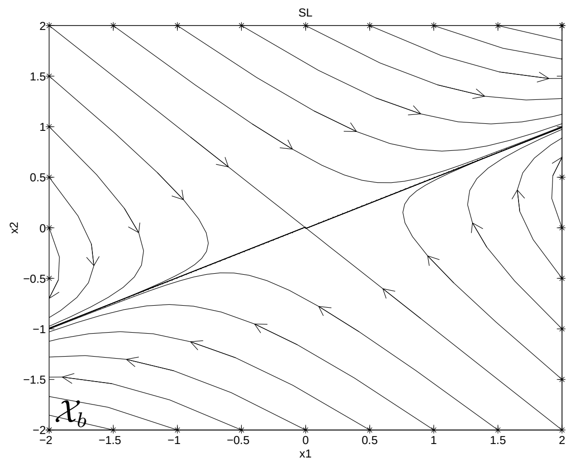
$$\lambda_1 = -2+j, \quad \lambda_2 = -2-j, \quad \lambda_3 = 1, \quad \lambda_4 = -3$$

$$\mathbf{v}_1=\begin{bmatrix} -1 \\ 1+j \\ -2j \\ 1 \end{bmatrix}, \quad \mathbf{v}_2=\begin{bmatrix} -1 \\ 1-j \\ 2j \\ 1 \end{bmatrix}, \quad \mathbf{v}_3=\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4=\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

b.1) Draw qualitatively the trajectories generated in the plane $\mathcal{X}_a$ by the two complex conjugate eigenvalues of the dynamic system:



b.2) Draw qualitatively the trajectories generated in the plane $\mathcal{X}_b$ by the two real eigenvalues of the dynamic system:



c.1) Write the type of trajectories:

○ Node? ⊗ Focus? ○ Saddle?

⊗ Stable? ○ Unstable?

c.2) Write the type of trajectories:

○ Node? ○ Focus? ⊗ Saddle?

○ Stable? ⊗ Unstable?

d.1) Provide a base $\mathcal{B}_a$ of plane $\mathcal{X}_a = \mathrm{Im}\mathcal{B}_a$:

$$\mathcal{B}_a=\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$$

d.2) Provide a base $\mathcal{B}_b$ of plane $\mathcal{X}_b = \mathrm{Im}\mathcal{B}_b$:

$$\mathcal{B}_b=\begin{bmatrix} -1 & 0 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. Relatively to a discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ that has at least an eigenvalue in the origin, it can be stated that:

⊗ if the system is completely observable then it is also completely constructable;
○ if the system is completely constructable then it is also completely observable;
○ if the system is completely controllable then it is also completely reachable;
⊗ if the system is completely reachable then it is also completely controllable;

⊗ the system can be asymptotically stable;

8. An “open loop” state estimator can be used

   ◯ if and only if the system is observable;
   ⊗ if and only if the system is asymptotically stable;
   ◯ if and only if the unstable part of the system is observable;
   ◯ if and only if the not-observable part of the system is asymptotically stable;

9. A system ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$) is “stabilizable” using a static state feedback

   ⊗ if the system is stable;
   ◯ if the system is observable;
   ⊗ if the unstable part of the system is reachable;
   ⊗ if the part not-reachable of the system is stable;

10. The Ackermann formula for computing the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of the feedback system can be used:

   ◯ for whatever system;
   ⊗ only if the system is reachable;
   ⊗ only for systems with one input;
   ⊗ also for unstable systems;

   Provide the explicit description of the Ackermann formula and the desired polynomial $p(\lambda)$ when all the n eigenvalues of the system must be located in $\lambda = -3$:

$$\mathbf{k}^{\mathrm{T}} = -\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} (\mathcal{R}^{+})^{-1} p(\mathbf{A}) \qquad\qquad p(\lambda) = (\lambda + 3)^n$$

11. Given a SISO linear system of the forth order ($n = 4$), completely reachable, characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$.
   a) Write the structure of the matrices $\mathbf{A}_c$, $\mathbf{b}_c$ and $\mathbf{c}_c$ of the corresponding controllability canonical form. Let $p(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ the characteristic polynomial of matrix $\mathbf{A}$.

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix}, \qquad \mathbf{b}_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{c}_c = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

   b) Moreover, write the structure of matrix $\mathbf{T}$ which, together with the space transformation $\mathbf{x} = \mathbf{T}\mathbf{x}_c$, brings the system in controllability canonical form.

$$\mathbf{T} = \mathcal{R}^{+}(\mathcal{R}_c^{+})^{-1} = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 1 \\ \alpha_2 & \alpha_3 & 1 & 0 \\ \alpha_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

12. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$ with period T:

$$\mathbf{F} = e^{\mathbf{A}T}, \qquad \mathbf{G} = \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\, d\sigma, \qquad \mathbf{H} = \mathbf{C}$$

13. Write the structure of the matrix $\mathbf{P}^{-1}$ of the state space transformation $\mathbf{x} = \mathbf{P}\overline{\mathbf{x}}$ which brings a not-observable system in the standard observability form:

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} \qquad \text{where} \qquad \text{Im}\mathbf{P}_1^{\mathrm{T}} = \text{Im}(\mathcal{O}^-)^{\mathrm{T}} \text{ and } \mathbf{P}_2 \text{ makes non singular the matrix } \mathbf{P}^{-1}.$$

Moreover, write the block structure of the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$:

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{1,1} & 0 \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix} \qquad \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & 0 \end{bmatrix}$$

Write the simplified form of transfer matrix $\mathbf{H}(s)$ of the system $\mathcal{S}$ as a function of the sub-matrices $\mathbf{A}_{i,j}$, $\mathbf{B}_i$ and $\mathbf{C}_j$ that characterize the system $\bar{\mathcal{S}} = (\overline{\mathbf{A}},\ \overline{\mathbf{B}},\ \overline{\mathbf{C}})$:

$$\mathbf{H}(s) = \mathbf{C}_1(s\,\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{B}_1$$

14. Write the "*separation property*" of the regulator:

The design of the feedback block $(\mathbf{A} + \mathbf{B}\mathbf{K})$ *and the estimation block* $(\mathbf{A} + \mathbf{L}\mathbf{C})$ *can be done independently:*

$$\det[z\mathbf{I} - \bar{\mathbf{A}}] = \det[z\mathbf{I} - (\mathbf{A} + \mathbf{B}\mathbf{K})]\,\det[z\mathbf{I} - (\mathbf{A} + \mathbf{L}\mathbf{C})]$$

15. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ that move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the structure of the solution $\mathbf{u}$ which minimizes the Euclidean norm $||\mathbf{u}||$:

$$\mathbf{u} = (\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}[\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0)]$$

16. Write the *Heymann Lemma*:
If $(\mathbf{A},\ \mathbf{B})$ *is reachable and if* $\mathbf{b}_i$ *is a not zero column of* $\mathbf{B}$, *then it exists a matrix* $\mathbf{M}_i \in \mathcal{R}^{m\times n}$ *such that* $(\mathbf{A} + \mathbf{B}\mathbf{M}_i,\ \mathbf{b}_i)$ *is reachable.*

17. Consider a continuous-time nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ and let $\mathbf{x}_0$ an equilibrium point of the system for constant input $\mathbf{u}_0$. Write the linear part of the Taylor series expansion of the function $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ in the neighborhood of point $(\mathbf{x}_0,\ \mathbf{u}_0)$:

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = \underbrace{\mathbf{f}(\mathbf{x}_0,\mathbf{u}_0)}_{0} + \left.\frac{\partial\mathbf{f}(\mathbf{x},\ \mathbf{u})}{\partial\,\mathbf{x}}\right|_{(\mathbf{x}_0,\mathbf{u}_0)}(\mathbf{x}-\mathbf{x}_0) + \left.\frac{\partial\mathbf{f}(\mathbf{x},\ \mathbf{u})}{\partial\,\mathbf{u}}\right|_{(\mathbf{x}_0,\mathbf{u}_0)}(\mathbf{u}-\mathbf{u}_0) + \mathbf{h}_1(\mathbf{x},\mathbf{u})$$

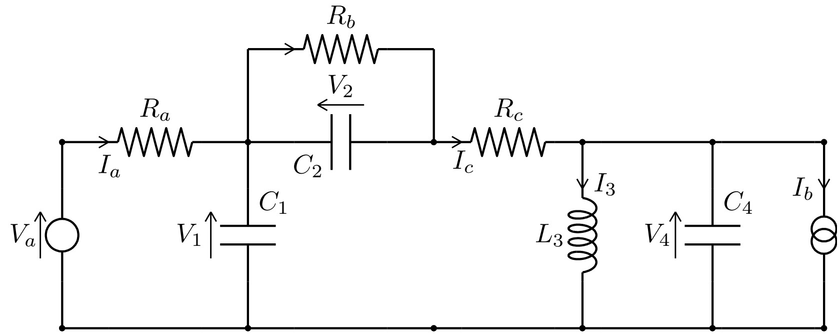
18. Compute the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ of the following *continuous-time* nonlinear system:

$$\left\{\begin{array}{rcl} \dot{x}_1(t) &=& x_2(t) \\ \dot{x}_2(t) &=& x_1(t)\,(x_1^2(t) - 1) - 2x_2(t) \end{array}\right. \qquad \underset{\dot{\mathbf{x}}=0}{\Rightarrow} \qquad \left\{\begin{array}{rcl} \overline{\mathbf{x}}_1 &=& (\ 0,\ 0\ ) \\ \overline{\mathbf{x}}_2 &=& (\ 1,\ 0\ ) \\ \overline{\mathbf{x}}_3 &=& (-1,\ 0\ ) \end{array}\right.$$

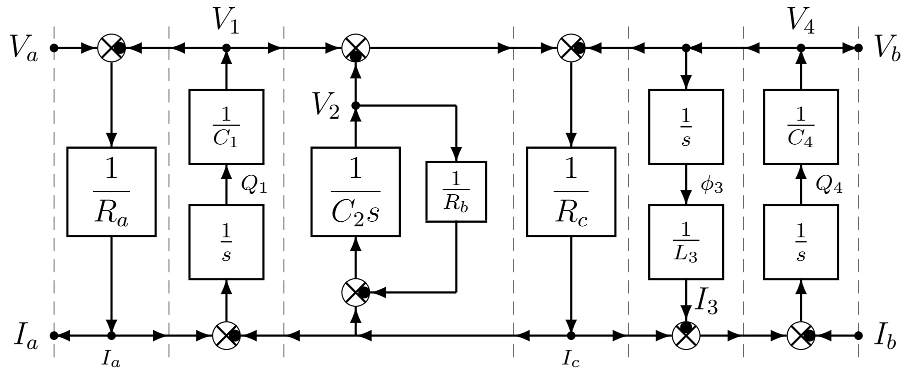
19. Compute the 3 equilibrium points $\tilde{\mathbf{x}}_1$, $\tilde{\mathbf{x}}_2$ and $\tilde{\mathbf{x}}_3$ of the following *discrete-time* nonlinear system:

$$\left\{\begin{array}{rcl} x_1(k+1) &=& x_2(k) \\ x_2(k+1) &=& x_1(k)\,(x_1^2(k) - 1) - 2x_2(k) \end{array}\right. \qquad \underset{\mathbf{x}(k+1)=\mathbf{x}(k)}{\Rightarrow} \qquad \left\{\begin{array}{rcl} \tilde{\mathbf{x}}_1 &=& (\ 0,\ \ 0\ ) \\ \tilde{\mathbf{x}}_2 &=& (\ 2,\ \ 2\ ) \\ \tilde{\mathbf{x}}_3 &=& (-2,-2\ ) \end{array}\right.$$

20. Consider the following electric circuit composed by the capacities C_1, C_2, C_4, the inductance L_3 and the resistances R_a, R_b and R_c. Two inputs act on the system: the voltage V_a and the current I_b. The outputs of the system are: the current I_a and the voltage V_b.



The POG model of the given electric circuit has the following structure:



Let $\mathbf{x} = \begin{bmatrix} V_1 & V_2 & I_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$
\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\overline{\mathbf{L}}}
\underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}}
=
\underbrace{\begin{bmatrix} -\frac{1}{R_a} - \frac{1}{R_c} & \frac{1}{R_c} & 0 & \frac{1}{R_c} \\ \frac{1}{R_c} & -\frac{1}{R_b} - \frac{1}{R_c} & 0 & -\frac{1}{R_c} \\ 0 & 0 & 0 & 1 \\ \frac{1}{R_c} & -\frac{1}{R_c} & -1 & -\frac{1}{R_c} \end{bmatrix}}_{\overline{\mathbf{A}}}
\underbrace{\begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}}
+
\underbrace{\begin{bmatrix} \frac{1}{R_a} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\overline{\mathbf{B}}}
\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}
$$

$$
\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}}
=
\underbrace{\begin{bmatrix} -\frac{1}{R_a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\overline{\mathbf{C}}}
\mathbf{x} +
\underbrace{\begin{bmatrix} \frac{1}{R_a} & 0 \\ 0 & 0 \end{bmatrix}}_{\overline{\mathbf{D}}}
\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}
$$

21. Write the energy variables q_1, q_2 and the output power variables v_1 and v_2 of the dynamic elements $\mathcal{D}_1$ and $\mathcal{D}_1$ that characterize the energetic domains:

	Electrical	**Mech. Trans.**	**Mech. Rot.**	**Hydraulic**
$\mathcal{D}_1$	C Capacity	M Mass	J Inertia	C_I Hydr. Capacity
q_1	Q Charge	p Momentum	p Ang. Momentum	V Volume
v_1	V Voltage	v Velocity	ω Ang. Velocity	P Pressure
$\mathcal{D}_2$	L Inductance	E Elasticity	E Tor. Elasticity	L_I Hydr. Inductance
q_2	ϕ Flux	x Displacement	θ Ang. Displacement	ϕ_I Hudr. Flux
v_2	I Current	F Force	τ Torque	Q Volume flow rate

22. Given the following discrete-time nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$:

$$\begin{cases} x_1(k+1) &= \alpha x_2(k) - x_1^2(k)x_2(k) \\ x_2(k+1) &= -\alpha x_1(k) + x_1(k)x_2^2(k) \end{cases}$$

It is easy to verify that the origin $\mathbf{x}_1 = (0,\ 0)$ is an equilibrium point for the system.

a) Compute the Jacobian $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ of the nonlinear system:

The Jacobian $\mathbf{A}(\mathbf{x})$ has the following structure:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} -2x_1x_2 & \alpha - x_1^2 \\ -\alpha + x_2^2 & 2x_1x_2 \end{bmatrix},$$

b) Compute the matrix $\mathbf{A}_1$ of the linearized system at the point $\mathbf{x}_1 = (0, 0)$:

The matrix $\mathbf{A}_1$ of the linearized system has the following structure:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$$

c) Study, for varying parameter α, the stability of the nonlinear discrete system in the neighborhood of point $\mathbf{x}_1 = (0,\ 0)$ using the reduced Lyapunov criterion:

The characteristic polynomial and the eigenvalues of matrix $\mathbf{A}_1$ are:

$$\Delta_{\mathbf{A}_1}(z) = z^2 + \alpha^2 = 0 \qquad \rightarrow \qquad z_{1,2} = \pm j\alpha$$

The considered system is a discrete-time system. Using the reduced Lyapunov criterion it can be stated that for $|\alpha| > 1$ the equilibrium point $\mathbf{x}_1 = (0, 0)$ of the nonlinear system is unstable, while for $|\alpha| < 1$ the equilibrium point $\mathbf{x}_1$ is asymptotically stable. For $|\alpha| = 1$ the criterion cannot be used.

d) For $\alpha = 1$, study the stability of the nonlinear system in the vicinity of the origin $\mathbf{x}_1 = (0, 0)$ using the "direct" Lyapunov criterion and the function: $V(\mathbf{x}(k)) = x_1^2(k) + x_2^2(k)$. Eventually, use the La Salle - Krasowskii criterion.

In the neighbourhood of the origin the function $V(\mathbf{x}(k)) = x_1^2(k) + x_2^2(k)$ is surely positive definite. Set $\alpha = 1$, the function $\Delta V(\mathbf{x}(k))$ computed along the system's trajectories is the following:

$$\begin{aligned}
\Delta V(\mathbf{x}(k)) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\
&= (x_2 - x_1^2 x_2)^2 + (-x_1 + x_1 x_2^2)^2 - x_1^2 - x_2^2 \\
&= x_2^2 + x_1^4 x_2^2 - 2x_1^2 x_2^2 + x_1^2 + x_1^2 x_2^4 - 2x_1^2 x_2^2 - x_1^2 - x_2^2 \\
&= x_1^4 x_2^2 - 4x_1^2 x_2^2 + x_1^2 x_2^4 = x_1^2 x_2^2(-4 + x_1^2 + x_2^2) \leq 0
\end{aligned}$$

The function $\Delta V(\mathbf{x}(k))$ is negative semidefinite and therefore, using the "direct" Lyapunov criterion, it can be stated that in the vicinity of the origin the nonlinear system is stable. The set $\mathcal{N} = \{(\alpha, 0), \alpha \in R\} \cup \{(0, \beta), \beta \in R\}$ of all the points that nullify the function $\Delta V(\mathbf{x}(k))$ contains perturbed trajectories of the system. In fact, when $(x_1, \; x_2) \in \mathcal{N}$ the given system simplifies as follows:

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = -x_1(k) \end{cases} \Rightarrow \quad \begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) \quad \Rightarrow \quad \mathbf{x}(k) = \mathbf{A}^k \mathbf{x}_0, \quad \text{where} \\ \mathbf{A}^k = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^k = \begin{bmatrix} \cos(\frac{k\pi}{2}) & \sin(\frac{k\pi}{2}) \\ -\sin(\frac{k\pi}{2}) & \cos(\frac{k\pi}{2}) \end{bmatrix} \end{cases}$$

For each $\mathbf{x}_0 \in \mathcal{N}$, the solution of the system is periodic and therefore it can be stated that for $\alpha = 1$ the nonlinear system is simply stable in the vicinity of the origin.