System and Control Theory
Test of December 20, 2012
Questions and Exercises

Name:	
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Signature:	

1. Write the general solution of the following time-varying difference equation $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$, being $\mathbf{x}(h)$ the state at time h:

   $$\mathbf{x}(k) =$$

   where $\mathbf{\Phi}(k,h)$ is the state transition matrix of the system.

2. Write the discrete time behavior of the output function $\mathbf{y}(t)$, solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and the static equation $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(0)$ at time $t_0 = 0$:

   $$\mathbf{y}(t) =$$

3. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \quad \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \quad \end{bmatrix}$$

   The system is: ○ reachable? ○ not-reachable? ○ observable? ○ not-observable?

   Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

   $$\mathcal{X}^+ = \mathrm{Im}\,[\mathcal{B}_R] = \mathrm{Im}\begin{bmatrix} \quad \end{bmatrix}, \qquad \mathcal{E}^- = \mathrm{Im}\,[\mathcal{B}_O] = \mathrm{Im}\begin{bmatrix} \quad \end{bmatrix}.$$

4. The following symbolic representation:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k) \end{cases}$$

   is used for describing a system with the following characteristics:

   ○ a linear system; ○ a time-varying system ;
   ○ a dynamic system; ○ a lumped system;
   ○ a continuous-time system; ○ an autonomous system;

5. Apply the $\mathcal{Z}$-transform to the following discrete system:

$$\mathcal{Z}\left[\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)\right], \qquad \mathcal{Z}\left[\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)\right]$$

   and provides the expression of the transformed function $\mathbf{y}(z)$ of the output vector $\mathbf{y}(k)$ as a function of the initial state $\mathbf{x}_0$ and of the transform $\mathbf{u}(z)$ of the input signal $\mathbf{u}(k)$:

   $$\mathbf{y}(z) =$$

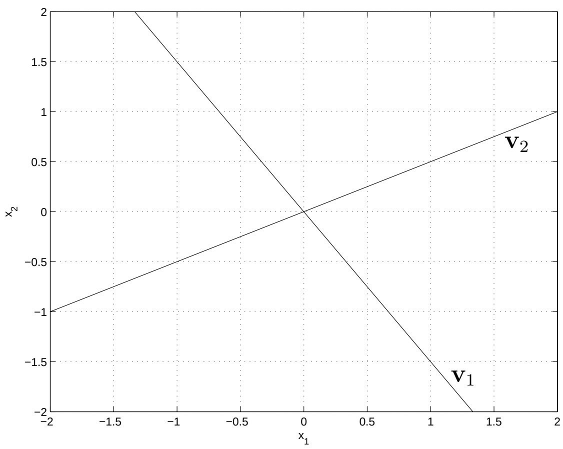
6. Draw the block scheme of the following continuous-time system where with $\mathbf{x}_c$ denotes the vector $\mathbf{x}_c = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$.

$$\begin{cases} \dot{\mathbf{x}}_c(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix} \mathbf{x}_c(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \mathbf{x}_c(t) \end{cases}$$

$$\boxed{\tfrac{1}{s}} \qquad \boxed{\tfrac{1}{s}} \qquad \boxed{\tfrac{1}{s}} \qquad \boxed{\tfrac{1}{s}}$$

7. Draw qualitatively the trajectories of a second order dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ characterized by the eigenvalues λ_i and the eigenvectors $\mathbf{v}_i$ shown in the two following boxes.

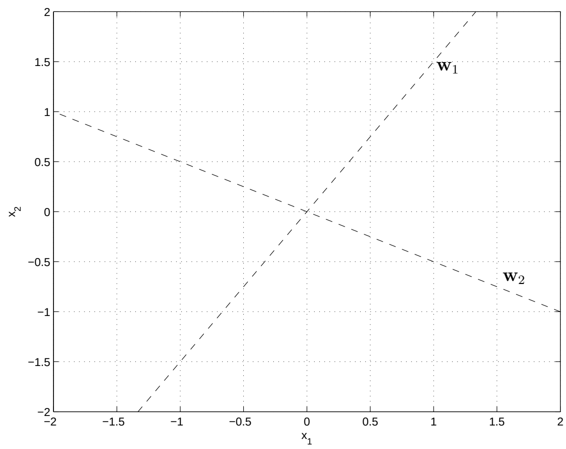
1) Eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -3$. The corresponding real eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are shown in figure.



○ Node? ○ Focus? ○ Saddle?

○ Stable? ○ Unstable?

2) Eigenvalues: $\lambda_{1,2} = 2 \pm 3j$. Eigenvectors: $\mathbf{v}_1 = \mathbf{w}_1 + \mathbf{w}_2 j$ and $\mathbf{v}_2 = \mathbf{v}_1^*$. The real vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are shown in figure.



○ Node? ○ Focus? ○ Saddle?

○ Stable? ○ Unstable?

8. Given the transfer function $G(z)$, write the structure of corresponding dynamic system in the reachability canonical form denoting with $u(k)$ the input and with $y(k)$ the output:

$$G(z) = \frac{z^4 + 6z^3 + 4z^2 + 7z + 4}{z^4 + 5z^3 + 2z^2 + 3z + 1}$$

$$\begin{cases} \mathbf{x}(k+1) &= \begin{bmatrix} \quad \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \quad \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} \quad \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \quad \end{bmatrix} u(t) \end{cases}$$

9. Consider the point-to-point control problem for a continuous-time linear system. Write the necessary condition that must be verified such that it esista a control law $\mathbf{u}$ that moves the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(t)$ in the time interval $[0,\ t]$:

$$\ldots$$

10. Compute the k-th power of the following Jordan miniblock:

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

11. Given a SISO linear system of the forth order, completely observable, characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$. Let $p(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ be the characteristic polynomial of matrix $\mathbf{A}$. Write the structure of matrix $\mathbf{P}$ which, together with the space transformation $\mathbf{x} = \mathbf{P}\mathbf{x}_o$, brings the system in the observability canonical form.

$$\mathbf{P} =$$

12. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of:

a) an **open loop** state estimator and the time evolution of the corresponding estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ starting from the initial condition $\mathbf{e}(0)$:

$\dot{\hat{\mathbf{x}}}(t) =$ $\qquad\qquad$ $\mathbf{e}(t) =$

b) a **in full order closed loop** state estimator and the time evolution of the corresponding estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ starting from the initial condition $\mathbf{e}(0)$:

$\dot{\hat{\mathbf{x}}}(t) =$ $\qquad\qquad$ $\mathbf{e}(t) =$

13. Relatively to the linear discrete system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)$, write a necessary and sufficient condition for the complete "constructability" of the system:

14. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$ with period T:

$\mathbf{F} =$ $\qquad\qquad$ $\mathbf{G} =$ $\qquad\qquad$ $\mathbf{H} =$

The corresponding sampled system is reachable if and only if for each couple λ_i, λ_j of eigenvalues of $\mathbf{A}$ having the same real part it is:

$$\ldots$$

15. Write the Ackermann formula for computing the gain vector $\mathbf{l}$ of an asymptotic state observer which freely places the eigenvalues of matrix $\mathbf{A} + \mathbf{lc}$:

$$\mathbf{l} =$$

This formula can be used

- ○ only for systems with one input;
- ○ only for systems which are completely observable;
- ○ for any type of time-invariant linear system;

16. Write the block matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$ of a system in the reachability standard form:

$$\overline{\mathbf{A}} = \left[\quad \right], \qquad \overline{\mathbf{B}} = \left[\quad \right], \qquad \overline{\mathbf{C}} = \left[\quad \right]$$

17. Given a discrete-time nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$. The equilibrium points of this system can be obtained :

- ○ computing the points $\mathbf{x}_e(k)$ tali that $\mathbf{x}_e(k) = \mathbf{f}(\mathbf{x}_e(k))$.
- ○ solving the equation $0 = \mathbf{f}(\mathbf{x}_e(k))$ with respect to the unknown $\mathbf{x}_e(k)$.
- ○ computing the points of the state space where $\mathbf{x}_e(k+1) = \mathbf{x}_e(k)$.

18. Write the structure of the dual system $\mathcal{S}_D$ corresponding to a given system $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$:

$$\mathcal{S}_D = (\quad , \quad , \quad , \quad)$$

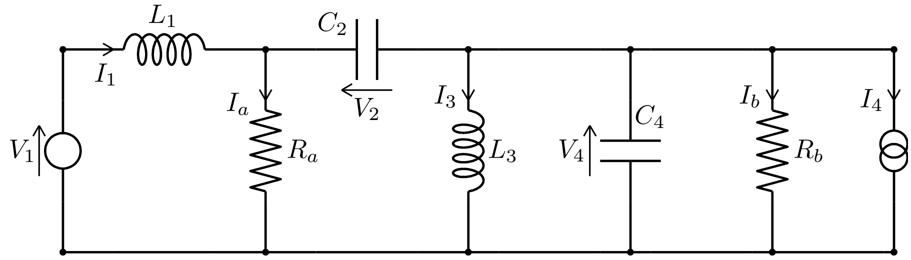
19. Consider a continuous-time nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$, $\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))$ and let $\mathbf{x}_0$ be an equilibrium point of the system for constant input $\mathbf{u}_0$. Write the definition of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ of the linearized system:

$\mathbf{A} =$ $\qquad$ $\mathbf{B} =$ $\qquad$ $\mathbf{C} =$ $\qquad$ $\mathbf{D} =$

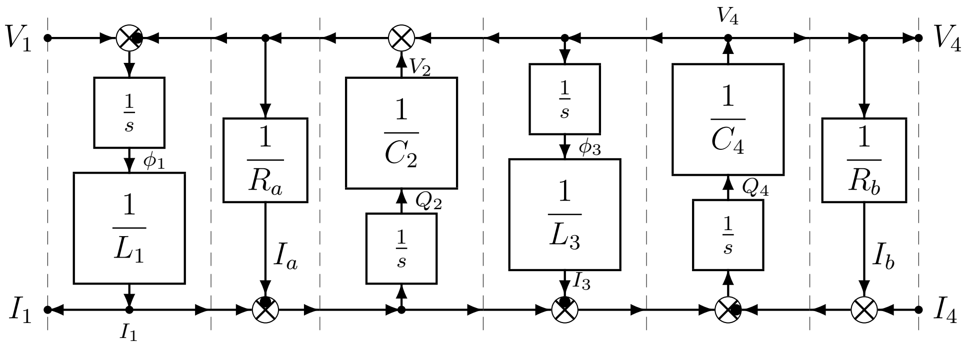
20. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Mechanical Rotational*. Moreover, write the constitutive relation (both linear and non linear) and the differential equation which characterize the physical elements:

	Symbols / Names	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$ q_1 v_1				
$\mathcal{D}_2$ q_2 v_2				
$\mathcal{R}$				

21. Consider the following electric circuit composed by the inductances L_1, L_3, the capacities C_2, C_4 and the resistances R_a and R_b. Two inputs act on the system: the voltage V_1 and the current I_4. The outputs of the system are: the current I_1 and the voltage V_4.



The POG model of the given electric circuit has the following structure:



Let $\mathbf{x} = \begin{bmatrix} I_1 & V_2 & I_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} V_1 & I_4 \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} I_1 & V_4 \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_1 \\ I_4 \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_1 \\ V_4 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \quad & \quad \\ & \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} V_1 \\ I_4 \end{bmatrix}}_{\mathbf{u}}$$

22. Write the La Salle - Krasowskii stability criterion for discrete-time nonlinear systems.

Consider the discrete-time nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.
If ...

23. Given the following continuous-time nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$:

$$\begin{cases} \dot{x}_1 &= x_2^2 - x_1^3 - x_1 \\ \dot{x}_2 &= \alpha x_2 - x_1^3 x_2 + u \end{cases}$$

Set $u = 0$, it is easy to verify that the origin $\mathbf{x}_1 = (0,\ 0)$ is an equilibrium point for the system.

a) Compute the matrix $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x},u)}{\partial \mathbf{x}}$ and the vector $\mathbf{b} = \frac{\partial \mathbf{f}(\mathbf{x},u)}{\partial u}$ of the nonlinear system:

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

b) Compute the matrix $\mathbf{A}_1$ and the vector $\mathbf{b}_1$ of the linearized system at the point $\mathbf{x}_1 = (0, 0)$, $u = 0$:

$$\mathbf{A}_1 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \qquad \mathbf{b}_1 = \begin{bmatrix} \quad \\ \quad \end{bmatrix},$$

c) Study, for varying parameter α, the stability of the nonlinear system in the neighborhood of point $\mathbf{x}_1 = (0,\ 0)$, $u = 0$ using the reduced Lyapunov criterion:

d) For $\alpha = 0$ and $u = 0$, study the stability of the nonlinear system in the vicinity of the origin $\mathbf{x}_1 = (0, 0)$ using the “direct” Lyapunov criterion and the function: $V(\mathbf{x}) = x_1^4 + 2x_2^2$. Eventually, use the La Salle - Krasowskii criterion.

e) Set $\alpha = 0$ and $u = k_1 x_1 + k_2 x_2$, compute for which values of the parameters k_1 and k_2 the linearized system in the neighborhood of point $\mathbf{x}_1 = (0, 0)$ is asymptotically stable.