

System and Control Theory
Test of January 26, 2012
Questions and Exercises

Name:	
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1. For a time-variant continuous-time linear system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$, the state transition matrix $\Phi(t, t_0)$ is solution of the following matrix differential equation:

$$\frac{d}{dt}\Phi(t, t_0) = \dots, \quad \Phi(t_0, t_0) = \dots$$

2. Write the explicit form of the *transition matrix* $\Phi(k, h)$ of a time-variant discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$:

$$\Phi(k, h) =$$

3. Write the closed form solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$ all' istante $t = t_0$:

$$\mathbf{x}(t) =$$

4. Compute the reachability matrix \mathcal{R}^+ and the observability matrix \mathcal{O}^- of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = [0 \ 0 \ 1] \mathbf{x}(t) \end{cases} \quad \mathcal{R}^+ = \begin{bmatrix} \\ \\ \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \\ \\ \end{bmatrix},$$

The system is: reachable? not-reachable? observable? not-observable?

Provide a base \mathcal{B}_R of the reachable subspace \mathcal{X}^+ and a base \mathcal{B}_O of the not-observable subspace \mathcal{E}^- :

$$\mathcal{X}^+ = \text{Im}[\mathcal{B}_R] = \text{Im} \begin{bmatrix} \\ \\ \end{bmatrix}, \quad \mathcal{E}^- = \text{Im}[\mathcal{B}_O] = \text{Im} \begin{bmatrix} \\ \\ \end{bmatrix}.$$

5. The following symbolic representation:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) \\ \mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k)) \end{cases} \quad \mathbf{x}(k) \in \mathbf{R}^n$$

is used for describing a system with the following characteristics:

- a dynamic system; a continuous-time system;
 a linear system; a lumped system;
 a time-varying system; a system without inputs;

6. Write the formula for computing the state transition matrix $e^{\mathbf{A}t}$ of a continuous-time linear system using the Laplace transform:

$$e^{\mathbf{A}t} =$$

7. Apply the \mathcal{Z} -transform to the following *state* function:

$$\mathcal{Z} [\mathbf{x}(k + 1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)]$$

and provides the expression of the transformed function $\mathbf{x}(z)$ of the state vector $\mathbf{x}(k)$ as a function of the initial state \mathbf{x}_0 and of the transform $\mathbf{u}(z)$ of the input signal $u(k)$:

$$\mathbf{x}(z) =$$

8. Given an autonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ of the fourth order where the matrix \mathbf{A} is given the “real” Jordan form $\mathbf{A}_R = \mathbf{T}_R^{-1}\mathbf{A}\mathbf{T}_R$ by using the state space transformation $\mathbf{x} = \mathbf{T}_R\bar{\mathbf{x}}$:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -2 & 0 \\ 0.5 & 2 & -0.5 & -1 \\ 2 & 0 & -1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_R = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T}_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

a) Write the eigenvalues λ_i and the eigenvectors \mathbf{v}_i that characterize the matrix \mathbf{A} :

$$\begin{aligned} \lambda_1 &= \\ \lambda_2 &= \\ \lambda_3 &= \\ \lambda_4 &= \end{aligned} \quad \mathbf{v}_1 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

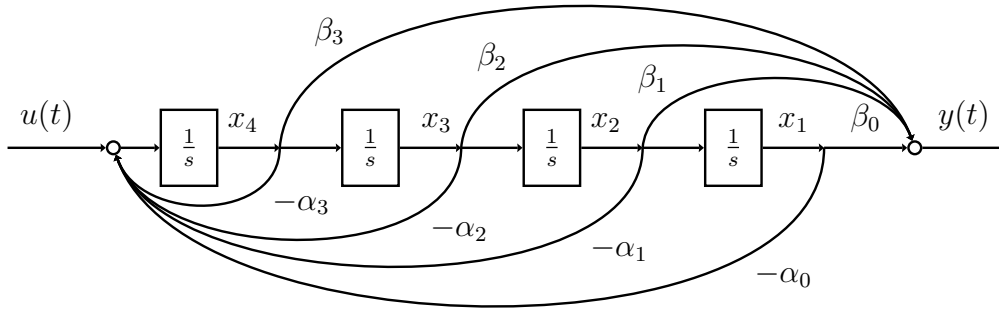
b) Write a transformation matrix \mathbf{T} (with $\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}}$) which brings the matrix \mathbf{A} in the Jordan diagonal form $\mathbf{A}_J = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$:

$$\mathbf{T} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \quad \mathbf{A}_J = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

c) Draw qualitatively the trajectories of the dynamic system:

<p>1) in the subspace \mathcal{X}_a generated by the two complex conjugate eigenvalues.</p> <div style="border: 1px solid black; width: 100%; height: 150px; margin: 10px 0;"></div> <p> <input type="radio"/> Node? <input type="radio"/> Focus? <input type="radio"/> Saddle? <input type="radio"/> Stable? <input type="radio"/> Unstable? </p>	<p>2) in the subspace \mathcal{X}_b generated by the two real eigenvalues.</p> <div style="border: 1px solid black; width: 100%; height: 150px; margin: 10px 0;"></div> <p> <input type="radio"/> Node? <input type="radio"/> Focus? <input type="radio"/> Saddle? <input type="radio"/> Stable? <input type="radio"/> Unstable? </p>
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9. Given following block scheme:



Set $\mathbf{x}_c = [x_1 \ x_2 \ x_3 \ x_4]^T$, write the structure of the matrices \mathbf{A} , \mathbf{B} and \mathbf{C} of a continuous-time system, in the state space, that describes the dynamics of the given block scheme.

$$\begin{cases} \dot{\mathbf{x}}_c(t) = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \mathbf{x}_c(t) + \begin{bmatrix} \\ \\ \\ \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} & & & \end{bmatrix} \mathbf{x}_c(t) \end{cases}$$

10. Given the following nonlinear differential equations in the state space:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -2x_2 \cos x_1 - 3x_1^2 + \sqrt{x_3} + u(t) \end{cases}$$

Set $[x_1 \ x_2 \ x_3]^T = [y(t) \ \dot{y}(t) \ \ddot{y}(t)]^T$, write the corresponding third order nonlinear differential equation which links the input $u(t)$ to the output $y(t)$:

...

11. For the discret-time linear systems $\mathbf{x}(k+1) = \mathbf{A} \mathbf{x}(k) + \mathbf{B} \mathbf{u}(k)$, write the condition that must be satisfied such that it is possible to move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0, k]$:

...

12. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0), x_2(0), x_3(0), x_4(0)]^T$, the free evolution of the following discrete-time autonomous system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{x}(k) \quad \mathbf{x}(k) = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

13. a) Write the explicit form of the Ackermann formula which provides the vector \mathbf{k}^T allowing the free positioning of the eigenvalues of a feedback system:

$$\mathbf{k}^T =$$

- b) Write the structure of the desired polynomial $p(\lambda)$ and the matrix $p(\mathbf{A})$ when $n = 5$, two systems's eigenvalues must be placed in $\lambda = -1$ e the other three systems's eigenvalues must be placed in $\lambda = -3$.

$$p(\lambda) =$$

$$p(\mathbf{A}) =$$

14. Write the structure of the matrix \mathbf{P}^{-1} of the state space transformation $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ which brings a not-observable system in the standard observability form:

$$\mathbf{P}^{-1} =$$

and \mathbf{P}_2 makes non singular the matrix \mathbf{P}

Moreover, write the block structure of the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$:

$$\bar{\mathbf{A}} = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} & \\ & \\ & \\ & \\ & \end{bmatrix}, \quad \bar{\mathbf{C}} = \begin{bmatrix} & & & & \end{bmatrix}$$

15. An “closed loop” state estimator can be used:

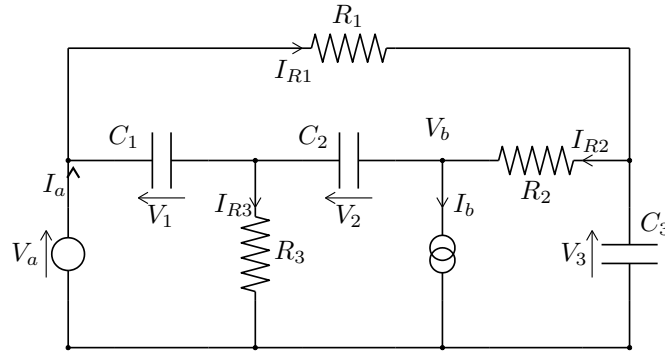
- if the system is observable;
- if the system is reachable;
- if the system is simply stable;
- if the unstable part of the system is observable;

16. Write the “*separation property*” of the regulator:

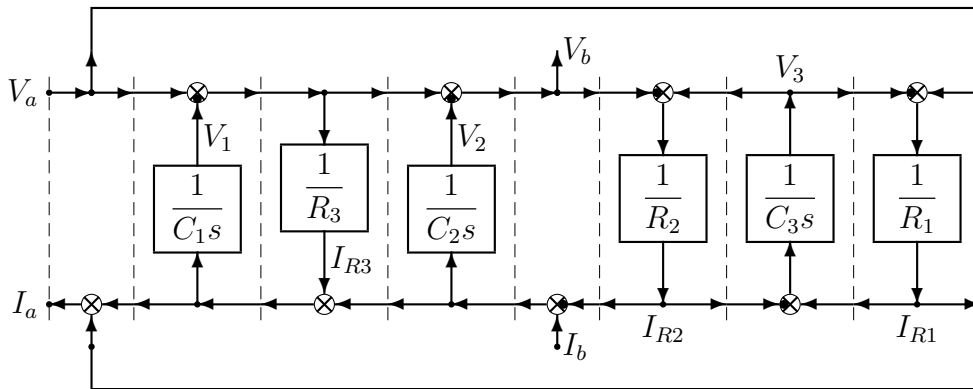
17. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Hydraulic*. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols / Names	Constitutive Rel.	Linear Case	Differential Eq.
\mathcal{D}_1				
q_1				
v_1				
\mathcal{D}_2				
q_2				
v_2				
\mathcal{R}				

18. Consider the following electric circuit (Stop Band Filter) composed by the capacities C_1 , C_2 , C_e and the resistances R_1 , R_2 ed R_2 . Two inputs act on the system: the voltage V_a and the current I_b . The outputs of the system are: the current I_a and the voltage V_b .



The POG model of the given electric circuit is the following:



Let $\mathbf{x} = [V_1 \ V_2 \ V_3]^T$ be the state vector, $\mathbf{u} = [V_a \ I_b]^T$ the input vector and $\mathbf{y} = [I_a \ V_b]^T$ the output vector. Write the corresponding dynamic system $\bar{\mathbf{L}}\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \bar{\mathbf{C}}\mathbf{x} + \bar{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \phantom{\dot{V}_1} \\ \phantom{\dot{V}_2} \\ \phantom{\dot{V}_3} \end{bmatrix}}_{\bar{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{\bar{\mathbf{A}}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \\ \end{bmatrix}}_{\bar{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \\ \end{bmatrix}}_{\bar{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \\ \end{bmatrix}}_{\bar{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

19. Write the direct Lyapunov stability criterion for continuous-time systems.

Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}_0)$ and let \mathbf{x}_0 be an equilibrium point corresponding to the constant input \mathbf{u}_0 .

1) If ...

20. Given the following nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, continuous-time and autonomous:

$$\begin{cases} \dot{x}_1 &= \alpha x_1^3 - \alpha x_2 \\ \dot{x}_2 &= x_1 - x_2^3 \end{cases}$$

a) Compute the position of the 3 equilibrium points $\bar{\mathbf{x}}_1$, $\bar{\mathbf{x}}_2$ and $\bar{\mathbf{x}}_3$ of the system:

$$\bar{\mathbf{x}}_1 = (\quad , \quad), \quad \bar{\mathbf{x}}_2 = (\quad , \quad), \quad \bar{\mathbf{x}}_3 = (\quad , \quad).$$

b) Compute the Jacobian $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ of the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

c) Compute the matrices \mathbf{A}_1 , \mathbf{A}_2 and \mathbf{A}_3 of the linearized system in the neighborhood of the 3 equilibrium points $\bar{\mathbf{x}}_1$, $\bar{\mathbf{x}}_2$ and $\bar{\mathbf{x}}_3$:

$$\mathbf{A}_1 = \mathbf{A}(\mathbf{x}_1) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \quad \mathbf{A}_2 = \mathbf{A}(\mathbf{x}_2) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \quad \mathbf{A}_3 = \mathbf{A}(\mathbf{x}_3) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

d) Study, for varying parameter α , the stability of the nonlinear system in the neighborhood of the 3 equilibrium points $\bar{\mathbf{x}}_1$, $\bar{\mathbf{x}}_2$ and $\bar{\mathbf{x}}_3$ using the reduced Lyapunov criterion:

21. Given the nonlinear discrete system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$ and the following function $V(\mathbf{x}(k))$:

$$\begin{cases} x_1(k+1) &= x_1 x_2 \\ x_2(k+1) &= x_1 - x_2 \end{cases} \quad V(\mathbf{x}(k)) = x_1^2 + x_2^2$$

Compute the function $\Delta V(\mathbf{x}(k))$ used in the direct Lyapunov criterion:

$$\Delta V(\mathbf{x}(k)) =$$