System and Control Theory
Test of January 26, 2012
Questions and Exercises

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1. For a time-variant continuous-time linear system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{x}(t)$, the state transition matrix $\mathbf{\Phi}(t, t_0)$ is solution of the following matrix differential equation:

$$\frac{d}{dt}\mathbf{\Phi}(t,t_0) = \mathbf{A}(t)\mathbf{\Phi}(t,t_0), \qquad \mathbf{\Phi}(t_0,t_0) = \mathbf{I}$$

2. Write the explicit form of the *transition matrix* $\mathbf{\Phi}(k,h)$ of a time-variant discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$:

$$\mathbf{\Phi}(k,h) = \begin{cases} \mathbf{A}(k-1)\ldots\mathbf{A}(h+1)\mathbf{A}(h) & \text{if } k > h \\ \mathbf{I} \text{ (Identity matrix)} & \text{if } k = h \end{cases}$$

3. Write the closed form solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$ all' istante $t = t_0$:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

4. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

The system is: ☒ reachable? ○ not-reachable? ☒ observable? ○ not-observable?

Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \text{Im}\,[\mathcal{B}_R] = \text{Im}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathcal{E}^- = \text{Im}\,[\mathcal{B}_O] = \text{Im}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

5. The following symbolic representation:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k),\mathbf{u}(k)) \\ \mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k),\mathbf{u}(k)) \end{cases} \qquad \mathbf{x}(k) \in \mathbf{R}^n$$

is used for describing a system with the following characteristics:

☒ a dynamic system;
○ a linear system;
○ a time-varying system;
○ a continuous-time system;
☒ a lumped system;
○ a system without inputs;

6. Write the formula for computing the state transition matrix $e^{\mathbf{A}t}$ of a continuous-time linear system using the Laplace transform:

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

7. Apply the $\mathcal{Z}$-transform to the following *state* function:

$$\mathcal{Z}\left[\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)\right]$$

and provides the expression of the transformed function $\mathbf{x}(z)$ of the state vector $\mathbf{x}(k)$ as a function of the initial state $\mathbf{x}_0$ and of the transform $\mathbf{u}(z)$ of the input signal $u(k)$:

$$\mathbf{x}(z) = (z\mathbf{I} - \mathbf{A})^{-1} z\, \mathbf{x}_0 + (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\, \mathbf{u}(z)$$

8. Given an autonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ of the forth order where the matrix $\mathbf{A}$ is given the "real" Jordan form $\mathbf{A}_R = \mathbf{T}_R^{-1}\mathbf{A}\mathbf{T}_R$ by using the state space transformation $\mathbf{x} = \mathbf{T}_R\bar{\mathbf{x}}$:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -2 & 0 \\ 0.5 & 2 & -0.5 & -1 \\ 2 & 0 & -1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{A}_R = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{T}_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

a) Write the eigenvalues λ_i and the eigenvectors $\mathbf{v}_i$ that characterize the matrix $\mathbf{A}$:

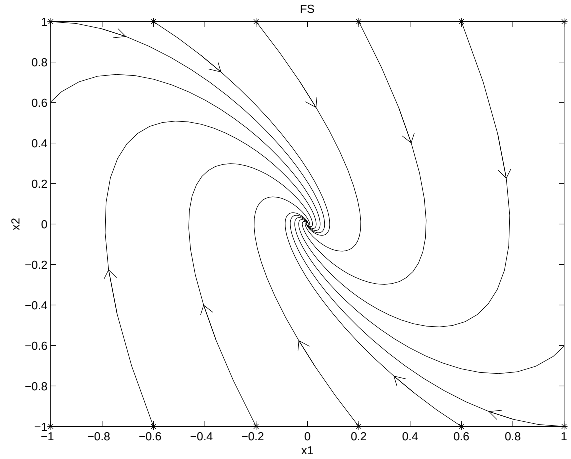
$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= -1+2j \\ \lambda_3 &= -1-2j \\ \lambda_4 &= 1 \end{aligned} \qquad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1+j \\ 0 \\ -1+j \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1-j \\ 0 \\ -1-j \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

b) Write a transformation matrix $\mathbf{T}$ (with $\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}}$) which brings the matrix $\mathbf{A}$ in the Jordan diagonal form $\mathbf{A}_J = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$:

$$\mathbf{T} = \begin{bmatrix} 0 & 1+j & 1-j & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1+j & -1-j & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{A}_J = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1+2j & 0 & 0 \\ 0 & 0 & -1+2j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) Draw qualitatively the trajectories of the dynamic system:

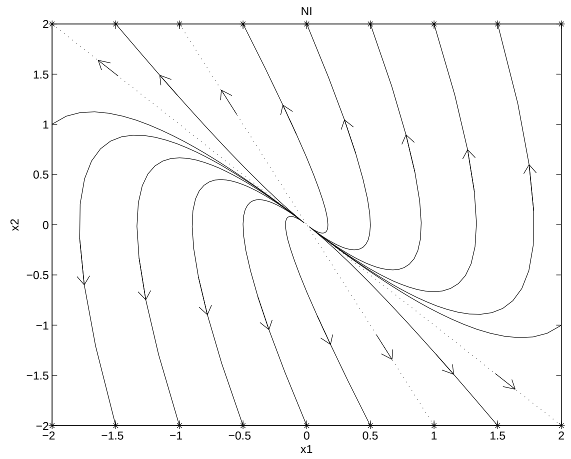
1) in the subspace $\mathcal{X}_a$ generated by the two complex conjugate eigenvalues.



○ Node? ☒ Focus? ○ Saddle?

☒ Stable? ○ Unstable?

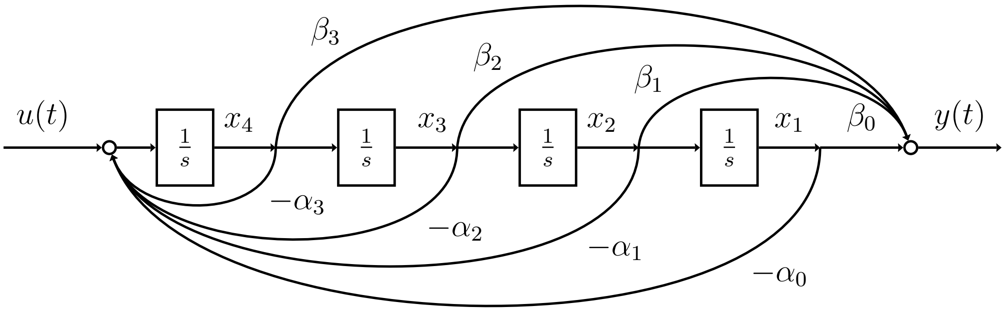
2) in the subspace $\mathcal{X}_b$ generated by the two real eigenvalues.



☒ Node? ○ Focus? ○ Saddle?

○ Stable? ☒ Unstable?

9. Given following block scheme:



Set $\mathbf{x}_c = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$, write the structure of the matrices **A**, **B** and **C** of a continuous-time system, in the state space, that describes the dynamics of the given block scheme.

$$\left\{\begin{array}{rcl} \dot{\mathbf{x}}_c(t) & = & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix} \mathbf{x}_c(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \mathbf{x}_c(t) \end{array}\right.$$

10. Given the following nonlinear differential equations in the state space:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & -2\,x_2 \cos x_1 - 3\,x_1^2 + \sqrt{x_3} + u(t) \end{array}\right.$$

Set $[x_1\ \ x_2\ \ x_3]^{\mathrm{T}} = [y(t)\ \ \dot{y}(t)\ \ \ddot{y}(t)]^{\mathrm{T}}$, write the corresponding third order nonlinear differential equation which links the input $u(t)$ to the output $y(t)$:

$$\dddot{y}(t) + 2\,\dot{y}(t)\cos y(t) + 3\,y(t)^2 - \sqrt{\ddot{y}(t)} = u(t).$$

11. For the discret-time linear systems $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,\mathbf{u}(k)$, write the condition that must be satisfied such that it is possible to move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$:

$$\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0) \in \mathcal{X}^+(k)$$

12. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^{\mathrm{T}}$, the free evolution of the following discrete-time autonomous system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{x}(k) \qquad \mathbf{x}(k) = \begin{bmatrix} 2^k & k\,2^{k-1} & \frac{k(k-1)}{2}2^{k-2} & 0 \\ 0 & 2^k & k\,2^{k-1} & 0 \\ 0 & 0 & 2^k & 0 \\ 0 & 0 & 0 & (-1)^k \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

13. a) Write the explicit form of the Ackermann formula which provides the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of a feedback system:

$$\mathbf{k}^{\mathrm{T}} = -\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} (\mathcal{R}^+)^{-1} p(\mathbf{A})$$

b) Write the structure of the desired polynomial $p(\lambda)$ and the matrix $p(\mathbf{A})$ when $n = 5$, two systems's eigenvalues must be placed in $\lambda = -1$ e the other three systems's eigenvalues must be placed in $\lambda = -3$.

$$p(\lambda) = (\lambda+1)^2(\lambda+3)^3, \qquad p(\mathbf{A}) = (\mathbf{A}+\mathbf{I})^2(\mathbf{A}+3\mathbf{I})^3$$

14. Write the structure of the matrix $\mathbf{P}^{-1}$ of the state space transformation $\mathbf{x} = \mathbf{P}\overline{\mathbf{x}}$ which brings a not-observable system in the standard observability form:

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} \qquad \text{where} \qquad \mathrm{Im}\mathbf{P}_1^{\mathrm{T}} = \mathrm{Im}(\mathcal{O}^{-})^{\mathrm{T}} \text{ and } \mathbf{P}_2 \text{ makes non singular the matrix } \mathbf{P}^{-1}.$$

Moreover, write the block structure of the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$:

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{1,1} & 0 \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix} \qquad \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & 0 \end{bmatrix}$$

15. An “closed loop” state estimator can be used:

☒ if the system is observable;
○ if the system is reachable;
○ if the system is simply stable;
☒ if the unstable part of the system is observable;

16. Write the “*separation property*” of the regulator:

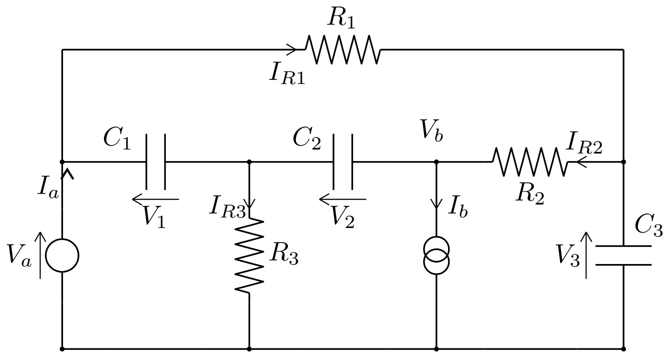
The design of feedback block $(\mathbf{A} + \mathbf{BK})$ *and the estimation block* $(\mathbf{A} + \mathbf{LC})$ *can be done independently:*

$$\det[z\mathbf{I} - \bar{\mathbf{A}}] = \det[z\mathbf{I} - (\mathbf{A} + \mathbf{BK})] \det[z\mathbf{I} - (\mathbf{A} + \mathbf{LC})]$$

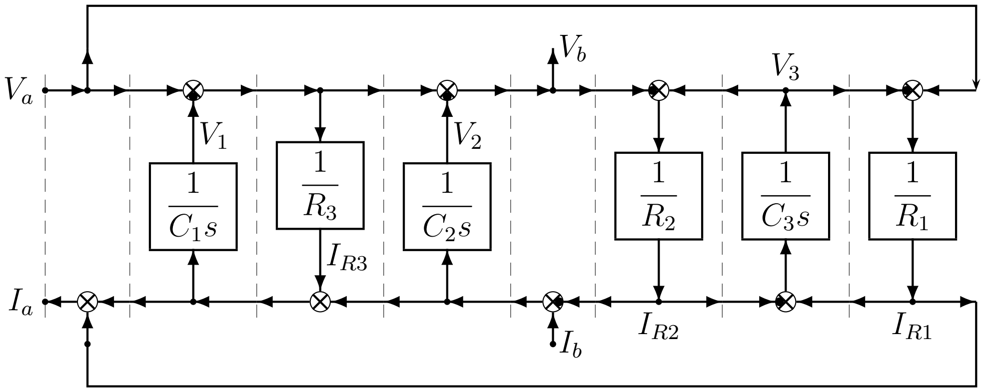
17. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Hydraulic*. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols		Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$	C_I	Hydr. Capacity	$V = \Phi_C(P)$	$V = C_I\, P$	$\dfrac{dV}{dt} = Q$
q_1	V	Volume			
v_1	P	Pressure			
$\mathcal{D}_2$	L_I	Hydr. Inductance	$\phi_I = \Phi_L(Q)$	$\phi_I = L_I\, Q$	$\dfrac{d\,\phi_I}{dt} = P$
q_2	ϕ_I	Hydr. Flux			
v_2	Q	Volume flow rate			
$\mathcal{R}$	R	Hydr. Resistance	$P = \Phi_R(Q)$	$P = R_I\, Q$	

18. Consider the following electric circuit (Stop Band Filter) composed by the capacities C_1, C_2, C_e and the resistances R_1, R_2 ed R_2. Two inputs act on the system: the voltage V_a and the current I_b. The outputs of the system are: the current I_a and the voltage V_b.



The POG model of the given electric circuit is the following:



Let $\mathbf{x} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -\frac{1}{R_3} - \frac{1}{R_2} & -\frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & -\frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & -\frac{1}{R_2} & -\frac{1}{R_2} - \frac{1}{R_1} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \frac{1}{R_3} + \frac{1}{R_2} & 1 \\ \frac{1}{R_2} & 1 \\ \frac{1}{R_2} + \frac{1}{R_1} & 0 \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -\frac{1}{R_3} - \frac{1}{R_2} & -\frac{1}{R_2} & -\frac{1}{R_2} - \frac{1}{R_1} \\ -1 & -1 & 0 \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} & 1 \\ 1 & 0 \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

19. Write the direct Lyapunov stability criterion for continuous-time systems.

Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.

1) If in a neighborhood W *of* $\mathbf{x}_0$ *it exists a function* $V(\mathbf{x}) : W \to \mathcal{R}$ positive definite *with continuous first time-derivatives and if* $\dot{V}(\mathbf{x})$ *is* negative semidefinite, *then the point* $\mathbf{x}_0$ *is* stable *for the nonlinear system .*

2) Moreover, if $\dot{V}(\mathbf{x})$ *is* negative definite, *then the point* $\mathbf{x}_0$ *is* asymptotically *stable.*

20. Given the following nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, continuous-time and autonomous:

$$\begin{cases} \dot{x}_1 &= \alpha x_1^3 - \alpha x_2 \\ \dot{x}_2 &= x_1 - x_2^3 \end{cases}$$

a) Compute the position of the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ of the system:

The equilibrium points of the system can be determined imposing $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$:

$$x_1^3 - x_2 = 0, \qquad\qquad x_1 - x_2^3 = 0.$$

The system has the following 3 equilibrium points:

$$\overline{\mathbf{x}}_1 = (0,\ 0), \qquad \overline{\mathbf{x}}_2 = (1,\ 1), \qquad \overline{\mathbf{x}}_3 = (-1,\ -1).$$

b) Compute the Jacobian $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ of the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$:

The Jacobian of the nonlinear system is:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} 3\alpha x_1^2 & -\alpha \\ 1 & -3x_2^2 \end{bmatrix}$$

c) Compute the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ of the linearized system in the neighborhood of the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$:

The matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_1$ of the linearized system have the following structure:

$$\mathbf{A}_1 = \mathbf{A}(\mathbf{x}_1) = \begin{bmatrix} 0 & -\alpha \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{A}_2 = \mathbf{A}(\mathbf{x}_2) = \begin{bmatrix} 3\alpha & -\alpha \\ 1 & -3 \end{bmatrix}, \qquad \mathbf{A}_3 = \mathbf{A}(\mathbf{x}_3) = \begin{bmatrix} 3\alpha & -\alpha \\ 1 & -3 \end{bmatrix}.$$

d) Study, for varying parameter α, the stability of the nonlinear system in the neighborhood of the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ using the reduced Lyapunov criterion:

The characteristic polynomial of matrix $\mathbf{A}_1$ is the following:

$$\Delta_{\mathbf{A}_1}(s) = s^2 + \alpha = 0 \qquad \rightarrow \qquad s_{1,2} = \pm j\sqrt{\alpha}$$

For $\alpha \geq 0$ the two eigenvalues of the system are on the imaginary axis and therefore the reduced Lyapunov criterion cannot be used. For $\alpha < 0$ at least one of the two eigenvalues is unstable and therefore also the equilibrium point $\mathbf{x}_1 = (0, 0)$ of the nonlinear system is unstable.

The characteristic polynomial of the matrices $\mathbf{A}_2$ and $\mathbf{A}_3$ is the following:

$$\Delta_{\mathbf{A}_2}(s) = \Delta_{\mathbf{A}_3}(s) = s^2 + 3(1 - \alpha)s - 8\alpha = 0$$

Using the reduced Lyapunov criterion it can be stated that: for $\alpha > 0$ the equilibrium points $\mathbf{x}_2 = (1, 1)$ and $\mathbf{x}_3 = (-1, -1)$ are unstable; for $\alpha = 0$ the criterion cannot be used; for $\alpha < 0$ the two equilibrium points are asymptotically stable.

21. Given the nonlinear discrete system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$ and the following function $V(\mathbf{x}(k))$:

$$\begin{cases} x_1(k+1) &= x_1 x_2 \\ x_2(k+1) &= x_1 - x_2 \end{cases} \qquad\qquad V(\mathbf{x}(k)) = x_1^2 + x_2^2$$

Compute the function $\Delta V(\mathbf{x}(k))$ used in the direct Lyapunov criterion:

$$\Delta V(\mathbf{x}(k)) = (x_1 x_2)^2 + (x_1 - x_2)^2 - x_1^2 - x_2^2 = x_1^2 x_2^2 - 2x_1 x_2.$$