System and Control Theory
Test of December 6, 2011
Questions and Exercises

Name:	
Nr. Mat.	
Signature:	

1. Write the explicit form of the *transition matrix* $\mathbf{\Phi}(k,h)$ of a time-invariant discrete-time linear system:

$$\mathbf{\Phi}(k,h) =$$

2. Write the general solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(0)$ at time $t_0 = 0$:

$$\mathbf{x}(t) =$$

3. Write the explicit solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ being $\mathbf{x}(h)$ the state at time h.

$$\mathbf{x}(k) =$$

4. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \quad \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \quad \end{bmatrix},$$

The system is: ○ reachable? ○ not-reachable? ○ observable? ○ not-observable?

Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \mathrm{Im}\,[\mathcal{B}_R] = \mathrm{Im} \begin{bmatrix} \quad \end{bmatrix}, \qquad \mathcal{E}^- = \mathrm{Im}\,[\mathcal{B}_O] = \mathrm{Im} \begin{bmatrix} \quad \end{bmatrix}.$$

5. The following symbolic representation:

$$\begin{cases} \dot{\mathbf{x}}(t) &=& \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) &=& \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \end{cases}$$

is used for describing a system with the following characteristics:

○ a static system; ○ a continuous-time system;
○ a linear system; ○ a lumped system;
○ a time-varying system; ○ a system without inputs;

6. Applying the state space transformation $\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}}$ to the dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ one obtains a transformed system $\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t)$, $\mathbf{y}(t) = \tilde{\mathbf{C}}\,\tilde{\mathbf{x}}(t)$ characterized by the following matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$:

$$\tilde{\mathbf{A}} = \qquad\qquad \tilde{\mathbf{B}} = \qquad\qquad \tilde{\mathbf{C}} =$$

7. Apply the Laplace transform to the following *state* function:

$$\mathcal{L}\left[\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right]$$

and provides the expression of the transformed function $\mathbf{x}(s)$ of the state vector $\mathbf{x}(t)$ as a function of the initial state $\mathbf{x}_0$ and of the transform function $\mathbf{u}(s)$ of the input signal $u(t)$:

$$\mathbf{x}(s) =$$

8. Given an autonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ of the forth order where the matrix $\mathbf{A}$ is characterized by the following eigenvalues λ_i, and eigenvectors $\mathbf{v}_i$:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 4 & 2 & -4 & -5 \\ 2 & 0 & -3 & -2 \\ -1 & 0 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = 2 \\ \lambda_3 = -2+j \\ \lambda_4 = -2-j \end{array} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ j \\ -1+j \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ -j \\ -1-j \\ 1 \end{bmatrix}.$$

a) Write a transformation matrix $\mathbf{T}$ (with $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$) that brings the matrix $\mathbf{A}$ in the Jordan diagonal form $\mathbf{A}_J$:

$$\mathbf{T} = \begin{bmatrix} \quad \end{bmatrix}, \qquad \mathbf{A}_J = \begin{bmatrix} \quad \end{bmatrix}$$

b) Write a transformation matrix $\mathbf{T}_R$ (with $\mathbf{x} = \mathbf{T}_R\bar{\mathbf{x}}$) that brings the matrix $\mathbf{A}$ in the "real" Jordan form $\mathbf{A}_R$:

$$\mathbf{T}_R = \begin{bmatrix} \quad \end{bmatrix}, \qquad \mathbf{A}_R = \begin{bmatrix} \quad \end{bmatrix}$$

c) Draw qualitatively the trajectories of the dynamic system:

1) in the subspace $\mathcal{X}_{1,2}$ generated by the two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.	2) in the subspace $\mathcal{X}_{3,4}$ generated by the two vectors Re[$\mathbf{v}_3$] and Im[$\mathbf{v}_3$].
○ Node? ○ Focus? ○ Saddle?	○ Node? ○ Focus? ○ Saddle?
○ Stable? ○ Unstable?	○ Stable? ○ Unstable?

9. Draw the block scheme of the following continuous-time system where $\mathbf{x}_o = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$.

$$\left\{ \begin{array}{rcl} \dot{\mathbf{x}}_o(t) & = & \begin{bmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{bmatrix} \mathbf{x}_o(t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_o(t) + d_0\, u(t) \end{array} \right.$$

10. Given the following nonlinear differential equation:

$$\dddot{y}(t) + 3\,\sin\ddot{y}(t) + 2\,\sqrt{\dot{y}(t)} + 5\,[y(t)]^3 = u(t).$$

Chosen $\mathbf{x} = [x_1\;\; x_2\;\; x_3]^{\mathrm{T}} = [y(t)\;\; \dot{y}(t)\;\; \ddot{y}(t)]^{\mathrm{T}}$ as state vector, write the corresponding nonlinear differential equation in the state space:

$$\begin{cases} \dot{x}_1 & = \\ \dot{x}_2 & = \\ \dot{x}_3 & = \end{cases}$$

11. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ that move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\; k]$ write the structure of the solution $\mathbf{u}$ which minimizes the Euclidean norm $||\mathbf{u}||$:

$$\mathbf{u} =$$

12. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and chosen T as sampling period, write the expression of matrix $\mathbf{F}$ that characterizes the corresponding sampled system $\mathbf{x}(k{+}1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{F} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

13. Given the transfer function $G(z)$, write the structure of corresponding dynamic system in the reachability canonical form denoting with $u(k)$ the input and with $y(k)$ the output:

$$G(z) = \frac{2z^3 + 4z^2 + 5z}{z^4 + 6z^3 + 3z^2 + 2z + 4} \qquad \begin{cases} \mathbf{x}(k{+}1) & = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix} u(k) \\ y(k) & = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \mathbf{x}(k) + [\quad] u(t) \end{cases}$$

14. Given the discrete-time linear system $\mathbf{x}(k{+}1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, write the structure of:

a) a *full order closed loop* state estimator:

$$\hat{\mathbf{x}}(k+1) =$$

b) the time evolution of the estimation error $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ obtained starting from the initial condition $\mathbf{e}(0)$:

$$\mathbf{e}(k) =$$

15. Given a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{u}$, time-invariant, completely reachable and with only one input. Let $\Delta_{\mathbf{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$ be the characteristic polynomial of matrix $\mathbf{A}$ and let $p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0$ be a monic polynomial freely chosen. Write the expression of vector $\mathbf{k}^{\mathrm{T}}$ which, using the static feedback $\mathbf{u} = \mathbf{k}^{\mathrm{T}}\mathbf{x}$, is able to match the eigenvalues of matrix $\mathbf{A} + \mathbf{b}\mathbf{k}^{\mathrm{T}}$ with the roots of polynomial $p(\lambda)$:

$$\mathbf{k}^{\mathrm{T}} =$$

where $\mathbf{k}_c^{\mathrm{T}} = [\qquad\qquad]$.

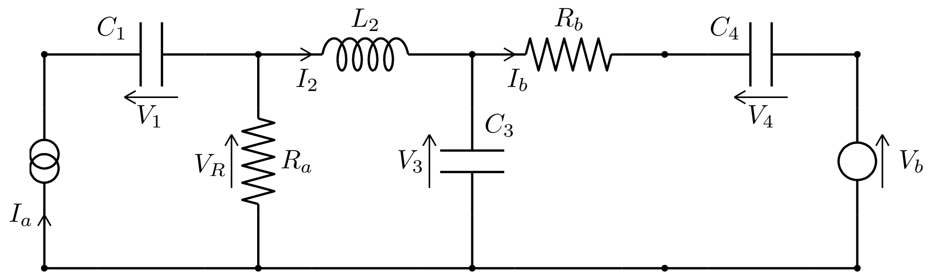
16. Write the block matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$ of a system in the reachability standard form:

$$\overline{\mathbf{A}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \qquad \overline{\mathbf{B}} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}, \qquad \overline{\mathbf{C}} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

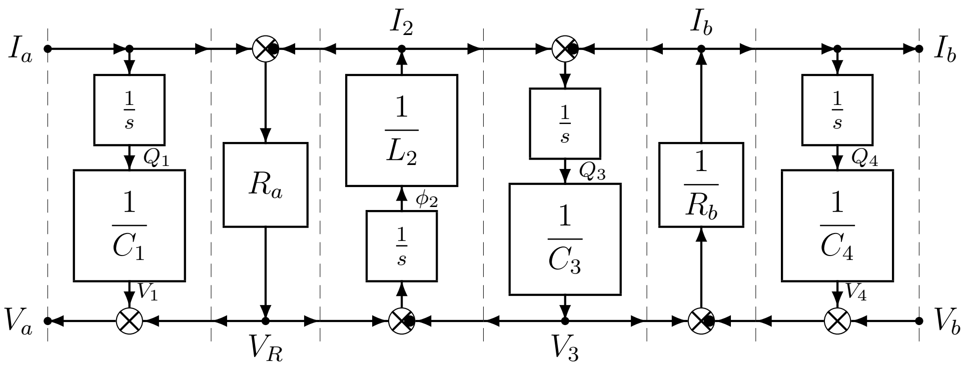
17. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Mechanical Translational*. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols / Names	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$ q_1 v_1				
$\mathcal{D}_2$ q_2 v_2				
$\mathcal{R}$				

18. Consider the following electric circuit composed by the capacities C_1, C_3, C_4, the inductance L_2 and the resistances R_a and R_b. Two inputs act on the system: the current I_a and the voltage V_b. The outputs of the system are: the voltage V_a and the current I_b.



The POG model of the given electric circuit is the following:



Let $\mathbf{x} = \begin{bmatrix} V_1 & I_2 & V_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} V_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \quad & \quad \\ \\ \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

19. Write the direct Lyapunov stability criterion for discrete-time nonlinear systems.

Consider the nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.
If ...

20. Given the following nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, continuous-time and autonomous:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1\,(x_1^2 - 1) - \alpha x_2 \end{cases}$$

a) Compute the position of the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ of the system:

$\overline{\mathbf{x}}_1 = (\quad,\quad)$, $\qquad \overline{\mathbf{x}}_2 = (\quad,\quad)$, $\qquad \overline{\mathbf{x}}_3 = (\quad,\quad)$.

b) Compute the Jacobian $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ of the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

c) Compute the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ of the linearized system in the neighborhood of the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$:

$$\mathbf{A}_1 = \mathbf{A}(\mathbf{x}_1) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \quad \mathbf{A}_2 = \mathbf{A}(\mathbf{x}_2) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}, \quad \mathbf{A}_3 = \mathbf{A}(\mathbf{x}_3) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

d) Study, for varying parameter α, the stability of the nonlinear system in the neighborhood of the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ using the reduced Lyapunov criterion:

e) For $\alpha = 0$, study the stability of the nonlinear system in the neighborhood of the equilibrium point $\mathbf{x}_1 = (0, 0)$ using the "direct" Lyapunov criterion and the following Lyapunov function: $V(\mathbf{x}) = x_1^2 - \frac{1}{2}x_1^4 + x_2^2$.