System and Control Theory
Test of December 6, 2011
Questions and Exercises

Name:	
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Signature:	

1. Write the explicit form of the *transition matrix* $\mathbf{\Phi}(k,h)$ of a time-invariant discrete-time linear system:

$$\mathbf{\Phi}(k,h) = \mathbf{A}^{k-h}$$

2. Write the general solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(0)$ at time $t_0 = 0$:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

3. Write the explicit solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ being $\mathbf{x}(h)$ the state at time h.

$$\mathbf{x}(k) = \mathbf{A}^{k-h}\mathbf{x}(h) + \sum_{j=h}^{k-1} \mathbf{A}^{k-j-1}\mathbf{B}\mathbf{u}(j)$$

4. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

The system is: ○ reachable? ⊗ not-reachable? ○ observable? ⊗ not-observable?

Provide a base $\mathcal{B}_R$ of the reachable subspace $\mathcal{X}^+$ and a base $\mathcal{B}_O$ of the not-observable subspace $\mathcal{E}^-$:

$$\mathcal{X}^+ = \mathrm{Im}\,[\mathcal{B}_R] = \mathrm{Im} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{E}^- = \mathrm{Im}\,[\mathcal{B}_O] = \mathrm{Im} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

5. The following symbolic representation:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \end{cases}$$

is used for describing a system with the following characteristics:

○ a static system;
○ a linear system;
⊗ a time-varying system;
⊗ a continuous-time system;
⊗ a lumped system;
○ a system without inputs;

6. Applying the state space transformation $\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}}$ to the dynamic system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ one obtains a transformed system $\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t)$, $\mathbf{y}(t) = \tilde{\mathbf{C}}\,\tilde{\mathbf{x}}(t)$ characterized by the following matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$:

$$\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \qquad \tilde{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}, \qquad \tilde{\mathbf{C}} = \mathbf{C}\mathbf{T}$$

7. Apply the Laplace transform to the following *state* function:

$$\mathcal{L}\left[\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right]$$

and provides the expression of the transformed function $\mathbf{x}(s)$ of the state vector $\mathbf{x}(t)$ as a function of the initial state $\mathbf{x}_0$ and of the transform function $\mathbf{u}(s)$ of the input signal $u(t)$:

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s)$$

8. Given an autonomous system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ of the forth order where the matrix $\mathbf{A}$ is characterized by the following eigenvalues λ_i, and eigenvectors $\mathbf{v}_i$:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 4 & 2 & -4 & -5 \\ 2 & 0 & -3 & -2 \\ -1 & 0 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = 2 \\ \lambda_3 = -2+j \\ \lambda_4 = -2-j \end{array} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ j \\ -1+j \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ -j \\ -1-j \\ 1 \end{bmatrix}.$$

a) Write a transformation matrix $\mathbf{T}$ (with $\mathbf{x} = \mathbf{T}\overline{\mathbf{x}}$) that brings the matrix $\mathbf{A}$ in the Jordan diagonal form $\mathbf{A}_J$:

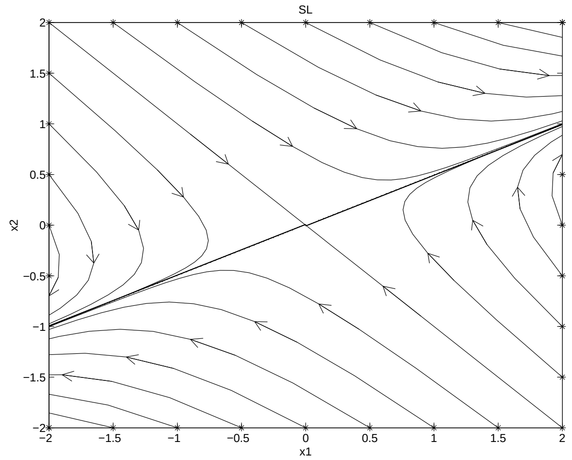
$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & j & -j \\ 1 & 0 & -1+j & -1-j \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{A}_J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2+j & 0 \\ 0 & 0 & 0 & -2-j \end{bmatrix}$$

b) Write a transformation matrix $\mathbf{T}_R$ (with $\mathbf{x} = \mathbf{T}_R\overline{\mathbf{x}}$) that brings the matrix $\mathbf{A}$ in the "real" Jordan form $\mathbf{A}_R$:

$$\mathbf{T}_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{A}_R = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

c) Draw qualitatively the trajectories of the dynamic system:

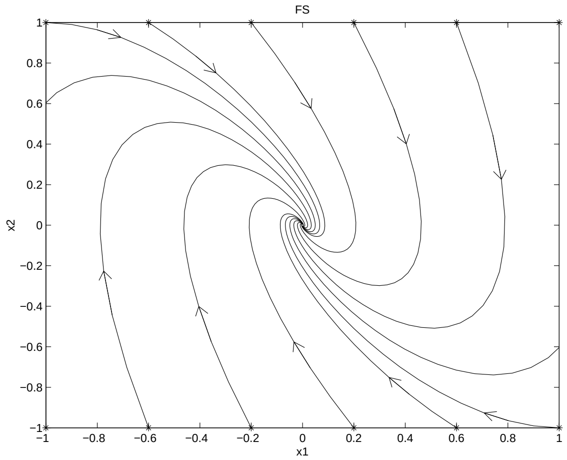
1) in the subspace $\mathcal{X}_{1,2}$ generated by the two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.



○ Node? ○ Focus? ⊗ Saddle?

○ Stable? ⊗ Unstable?

2) in the subspace $\mathcal{X}_{3,4}$ generated by the two vectors Re[$\mathbf{v}_3$] and Im[$\mathbf{v}_3$].

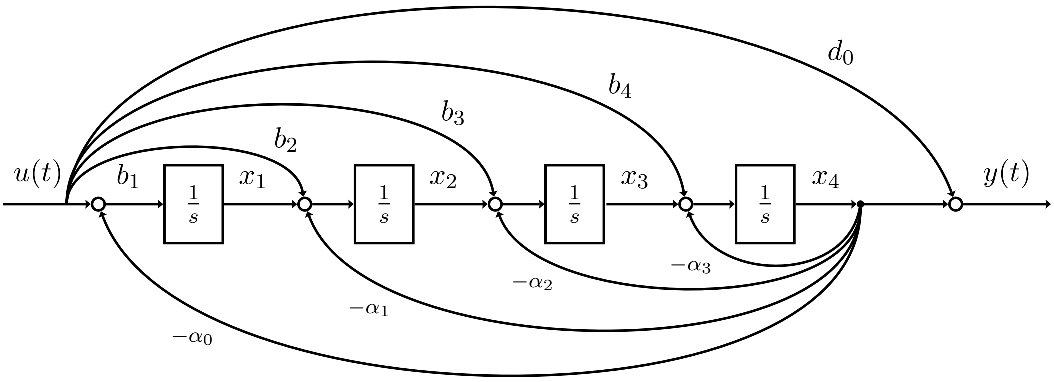


○ Node? ⊗ Focus? ○ Saddle?

⊗ Stable? ○ Unstable?

9. Draw the block scheme of the following continuous-time system where $\mathbf{x}_o = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$.

$$\left\{\begin{array}{rcl} \dot{\mathbf{x}}_o(t) & = & \begin{bmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{bmatrix} \mathbf{x}_o(t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_o(t) \; + \; d_0\, u(t) \end{array}\right.$$



10. Given the following nonlinear differential equation:

$$\dddot{y}(t) + 3\,\sin\ddot{y}(t) + 2\,\sqrt{\dot{y}(t)} + 5\,[y(t)]^3 = u(t).$$

Chosen $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^{\mathrm{T}} = [y(t) \;\; \dot{y}(t) \;\; \ddot{y}(t)]^{\mathrm{T}}$ as state vector, write the corresponding nonlinear differential equation in the state space:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & -3\,\sin x_3 - 2\,\sqrt{x_2} - 5\,x_1^3 + u(t) \end{array}\right.$$

11. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ that move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the structure of the solution $\mathbf{u}$ which minimizes the Euclidean norm $||\mathbf{u}||$:

$$\mathbf{u} = (\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}[\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0)]$$

12. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and chosen T as sampling period, write the expression of matrix $\mathbf{F}$ that characterizes the corresponding sampled system $\mathbf{x}(k{+}1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{F} = e^{\mathbf{A}T} = \begin{bmatrix} e^{-T} & T\,e^{-T} & \frac{T^2}{2}\,e^{-T} \\ 0 & e^{-T} & T\,e^{-T} \\ 0 & 0 & e^{-T} \end{bmatrix}$$

13. Given the transfer function $G(z)$, write the structure of corresponding dynamic system in the reachability canonical form denoting with $u(k)$ the input and with $y(k)$ the output:

$$G(z) = \frac{2z^3 + 4z^2 + 5z}{z^4 + 6z^3 + 3z^2 + 2z + 4} \qquad \left\{\begin{array}{l} \mathbf{x}(k{+}1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -2 & -3 & -6 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k) \\ \\ y(k) = \begin{bmatrix} 0 & 5 & 4 & 2 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \end{bmatrix} u(k) \end{array}\right.$$

14. Given the discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, write the structure of:

a) a *full order closed loop* state estimator:

$$\hat{\mathbf{x}}(k+1) = (\mathbf{A} + \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{L}\mathbf{y}(k)$$

b) the time evolution of the estimation error $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ obtained starting from the initial condition $\mathbf{e}(0)$:

$$\mathbf{e}(t) = (\mathbf{A} + \mathbf{L}\mathbf{C})^k\,\mathbf{e}(0)$$

15. Given a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$, time-invariant, completely reachable and with only one input. Let $\Delta_{\mathbf{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$ be the characteristic polynomial of matrix $\mathbf{A}$ and let $p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0$ be a monic polynomial freely chosen. Write the expression of vector $\mathbf{k}^{\mathrm{T}}$ which, using the static feedback $\mathbf{u} = \mathbf{k}^{\mathrm{T}}\mathbf{x}$, is able to match the eigenvalues of matrix $\mathbf{A} + \mathbf{b}\mathbf{k}^{\mathrm{T}}$ with the roots of polynomial $p(\lambda)$:

$$\mathbf{k}^{\mathrm{T}} = \mathbf{k}_c^{\mathrm{T}} \left\{ \begin{bmatrix} \mathbf{b}, & \mathbf{A}\mathbf{b}, & \ldots, & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & 1 \\ \alpha_2 & \ldots & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \right\}^{-1}$$

where $\mathbf{k}_c^{\mathrm{T}} = \begin{bmatrix} \alpha_0 - d_0, & \alpha_1 - d_1, & \ldots, & \alpha_{n-1} - d_{n-1} \end{bmatrix}$.

16. Write the block matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$ of a system in the reachability standard form:

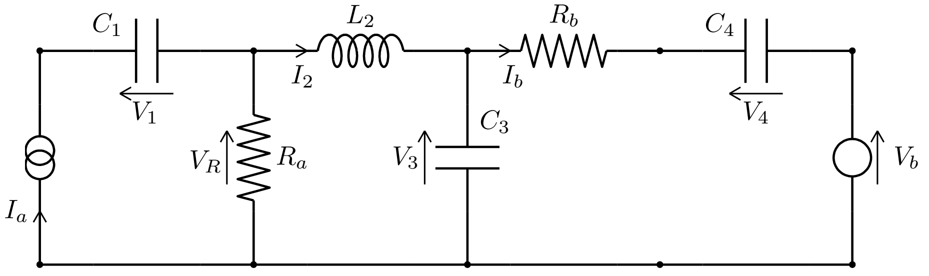
$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ 0 & \mathbf{A}_{2,2} \end{bmatrix} \qquad \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}$$

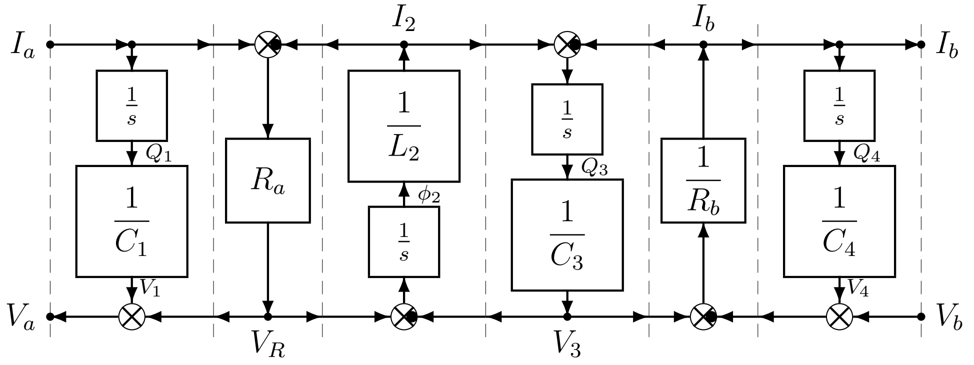
17. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Mechanical Translational*. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols		Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$	M	Mass	$P = \Phi_M(\dot{x})$	$P = M\,\dot{x}$	$\frac{d\,P}{dt} = F$
q_1	P	Momentum			
v_1	$\dot{x}$	Velocity			
$\mathcal{D}_2$	E	Elasticity	$x = \Phi_E(F)$	$x = E\,F$	$\frac{d\,x}{dt} = \dot{x}$
q_2	x	Displacement			
v_2	F	Force			
$\mathcal{R}$	b	Friction	$F = \Phi_b(\dot{x})$	$F = b\,\dot{x}$	

18. Consider the following electric circuit composed by the capacities C_1, C_3, C_4, the inductance L_2 and the resistances R_a and R_b. Two inputs act on the system: the current I_a and the voltage V_b. The outputs of the system are: the voltage V_a and the current I_b.



The POG model of the given electric circuit is the following:



Let $\mathbf{x} = \begin{bmatrix} V_1 & I_2 & V_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{V}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -R_a & -1 & 0 \\ 0 & 1 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} V_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ R_a & 0 \\ 0 & \frac{1}{R_b} \\ 0 & -\frac{1}{R_b} \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & -R_a & 0 & 0 \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} R_a & 0 \\ 0 & -\frac{1}{R_b} \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

19. Write the direct Lyapunov stability criterion for discrete-time nonlinear systems.

Consider the nonlinear system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.

If in a neighborhood W *of point* $\mathbf{x}_0$ *it exists a continuous function* $V(\mathbf{x}) : W \to \mathcal{R}$ positive definite *and if the function* $\Delta V(\mathbf{x})$ *is* negative semidefinite, *then the point* $\mathbf{x}_0$ *is* stable. *If the function* $\Delta V(\mathbf{x})$ *is* negative definite, *then the point* $\mathbf{x}_0$ *is* asymptotically stable.

20. Given the following nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, continuous-time and autonomous:

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_1\,(x_1^2 - 1) - \alpha x_2 \end{array}\right.$$

a) Compute the position of the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ of the system:

The equilibrium points of the system can be determined imposing $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$:

$$x_2 = 0, \qquad\qquad x_1\left(x_1^2 - 1\right) = 0.$$

The system has the following 3 equilibrium points:

$$\overline{\mathbf{x}}_1 = (0,\ 0), \qquad \overline{\mathbf{x}}_2 = (1,\ 0), \qquad \overline{\mathbf{x}}_3 = (-1,\ 0).$$

b) Compute the Jacobian $\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ of the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$:

The Jacobian of the nonlinear system is:

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 3x_1^2 - 1 & -\alpha \end{bmatrix}$$

c) Compute the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ of the linearized system in the neighborhood of the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$:

The matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_1$ of the linearized system have the following structure:

$$\mathbf{A}_1 = \mathbf{A}(\mathbf{x}_1) = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha \end{bmatrix}, \qquad \mathbf{A}_2 = \mathbf{A}(\mathbf{x}_2) = \begin{bmatrix} 0 & 1 \\ 2 & -\alpha \end{bmatrix}, \qquad \mathbf{A}_3 = \mathbf{A}(\mathbf{x}_3) = \begin{bmatrix} 0 & 1 \\ 2 & -\alpha \end{bmatrix}.$$

d) Study, for varying parameter α, the stability of the nonlinear system in the neighborhood of the 3 equilibrium points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ using the reduced Lyapunov criterion:

The characteristic polynomial of matrix $\mathbf{A}_1$ is the following:

$$\Delta_{\mathbf{A}_1}(s) = s^2 + \alpha s + 1 = 0$$

Using the reduced Lyapunov criterion it follows that: for $\alpha > 0$ the equilibrium point $\mathbf{x}_1 = (0, 0)$ of the nonlinear system is asymptotically stable: for $\alpha < 0$ the equilibrium point $\mathbf{x}_1$ is asymptotically stable; for $\alpha = 0$ the criterion cannot be used.

The characteristic polynomial of the matrices $\mathbf{A}_2$ and $\mathbf{A}_3$ is the following:

$$\Delta_{\mathbf{A}_2}(s) = \Delta_{\mathbf{A}_3}(s) = s^2 + \alpha s - 2 = 0$$

Using the reduced Lyapunov criterion it can be stated that the equilibrium points $\mathbf{x}_2 = (1, 0)$ and $\mathbf{x}_3 = (-1, 0)$ are both unstable for all the values of α.

e) For $\alpha = 0$, study the stability of the nonlinear system in the neighborhood of the equilibrium point $\mathbf{x}_1 = (0, 0)$ using the "direct" Lyapunov criterion and the following Lyapunov function: $V(\mathbf{x}) = x_1^2 - \frac{1}{2}x_1^4 + x_2^2$.

In the neighborhood of point $\mathbf{x}_1 = (0, 0)$ the function $V(\mathbf{x}) = x_1^2 - \frac{1}{2}x_1^4 + x_2^2$ is surely positive definite. Computing the time derivative of function $V(\mathbf{x})$ along the system's trajectories when $\alpha = 0$ one obtains:

$$\dot{V} = 2x_1\dot{x}_1 - 2x_1^3\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1x_2 - 2x_1^3x_2 + 2x_2x_1(x_1^2 - 1) = 0$$

Applying the "direct" Lyapunov criterion it can be stated that in the neighborhood of point $\mathbf{x}_1 = (0, 0)$ the nonlinear system is simply stable. In this case the trajectories of the system are equal to the "level curves" of function $V(\mathbf{x})$.