System and Control Theory
Test of January 25, 2011
Questions and Exercises

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1. Write the general solution of the linear time-variant differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$ being $\mathbf{x}(t_0)$ the state at time t_0:

$$\mathbf{x}(t) =$$

2. Write the explicit form of the *transition matrix* $\mathbf{\Phi}(k,h)$ of a discrete time-variant linear system $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$:

$$\mathbf{\Phi}(k,h) =$$

3. Write the explicit solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ being $\mathbf{x}(h)$ the state at time h.

$$\mathbf{x}(k) =$$

4. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix},$$

5. Given a linear POG dynamic system $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ where $\mathbf{L}$ is a symmetric positive definite matrix. The transfer matrix $\mathbf{H}(s)$ which links the input vector $\mathbf{U}(s)$ to the vector d'uscita $\mathbf{Y}(s)$ can be expressed as follows:

   ○ $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$
   ○ $\mathbf{H}(s) = \mathbf{C}(s\mathbf{L} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$
   ○ $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{L}^{-1}\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$
   ○ $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{L}^{-1}\mathbf{A})^{-1}\mathbf{L}^{-1}\mathbf{B} + \mathbf{D}$

6. A matrix $\mathbf{A}$ of dimension n is diagonalizable

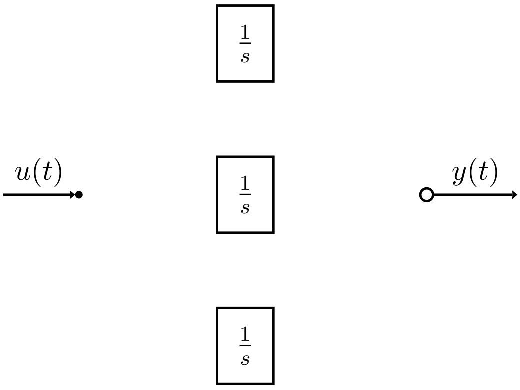
   ○ if it has n real distinct eigenvalues;
   ○ if it has n linearly independent eigenvectors;
   ○ if all the Jordan miniblocks have dimension equal to 1;
   ○ if all the eigenvalues have multiplicity greater than 1;

7. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ that move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the structure of the solution $\mathbf{u}$ which minimizes the Euclidean norm $||\mathbf{u}||$:

$$\mathbf{u} =$$

8. Draw the block scheme of the following continuous-time system in the Jordan canonical form where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}}$.

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \mathbf{x}(t) \end{cases}$$



9. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^{\mathrm{T}}$, the free evolution of the following continuous-time autonomous system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \mathbf{x}(t) \qquad \mathbf{x}(t) = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

10. Given the transfer function $G(s)$, write the structure of corresponding dynamic system in the reachability canonical form denoting with $u(t)$ the input and with $y(t)$ the output:

$$G(s) = 2 + \frac{3s^3 + 6s^2 + 2s + 4}{s^4 + 2s^3 + 5s^2 + 3s + 7} \qquad \begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \quad \end{bmatrix} u(t) \end{cases}$$

11. Given a SISO linear system of the forth order ($n = 4$), completely observable, characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$.
a) Write the structure of the matrices $\mathbf{A}_o$, $\mathbf{b}_o$ and $\mathbf{c}_o$ of the corresponding observability canonical form. Let $p(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ the characteristic polynomial of matrix $\mathbf{A}$.

$$\mathbf{A}_o = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}, \quad \mathbf{b}_o = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}, \quad \mathbf{c}_o = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}$$

b) Moreover, write the structure of matrix $\mathbf{P}$ which, together with the space transformation $\mathbf{x} = \mathbf{P}\mathbf{x}_o$, brings the system in the observability canonical form.

$$\mathbf{P} =$$

12. Given a continuous-time SISO lieanr system: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t)$, $y(t) = \mathbf{c}\,\mathbf{x}(t)$ with $\mathbf{A} \in \mathbf{R}^{n\times n}$, $\mathbf{b} \in \mathbf{R}^{n\times 1}$ and $\mathbf{c} \in \mathbf{R}^{1\times n}$. Write the number N of non-constant parameters a_{ij}, b_i and c_j which characterize each canonical form of the given system:

$$N =$$

13. Write the *Heymann Lemma*:

14. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of:

    a) a *full order closed loop* state estimator:

    $$\dot{\hat{\mathbf{x}}}(t) =$$

    b) the time evolution of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ obtained starting from the initial condition $\mathbf{e}(0)$:

    $$\mathbf{e}(t) =$$

15. Write the structure of the matrix $\mathbf{P}^{-1}$ of the state space transformation $\mathbf{x} = \mathbf{P}\overline{\mathbf{x}}$ which brings a not-observable system in the standard observability form:

    $$\mathbf{P}^{-1} =$$

    Moreover, write the block structure of the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$:

    $$\overline{\mathbf{A}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}, \qquad \overline{\mathbf{B}} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \qquad \overline{\mathbf{C}} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

16. Relatively to the linear discrete system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)$, a necessary and sufficient condition for the complete "constructability" of the system:

17. Write the direct Lyapunov stability criterion for continuous-time systems.

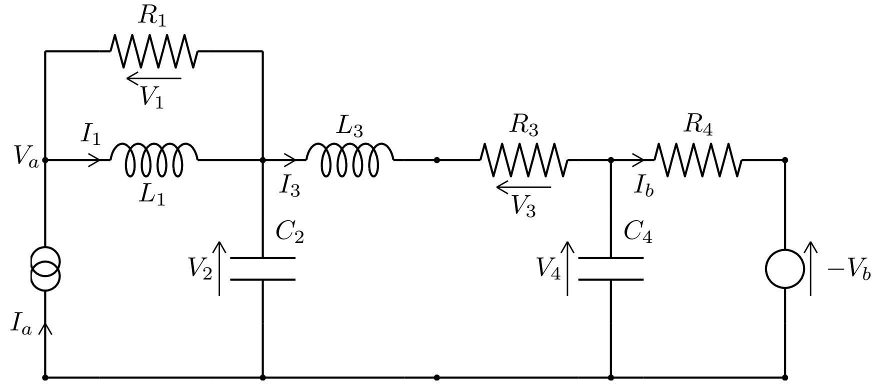
    Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$.

    1) If . . .

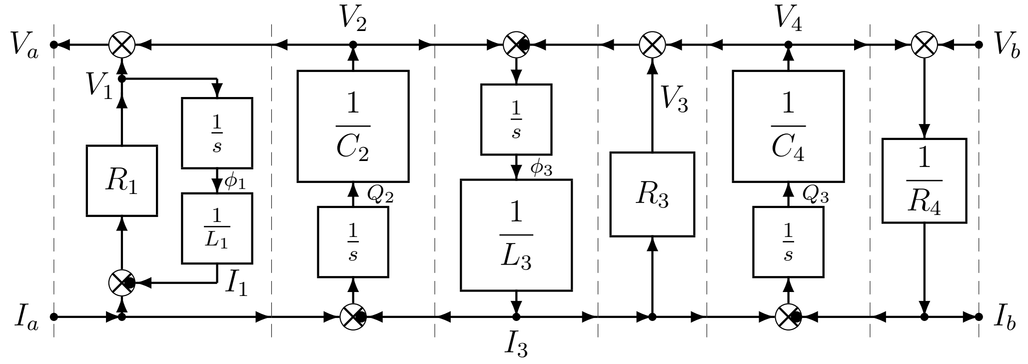
18. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$:

$$\mathbf{F} = \qquad\qquad\qquad \mathbf{G} = \qquad\qquad\qquad\qquad \mathbf{H} =$$

19. Consider the following electric circuit composed by the inductances L_1, L_3, the capacities C_2, C_3 and the resistances R_1, R_3 and R_4. Two inputs act on the system: the current I_a and the voltage V_b. The outputs of the system are: the voltage V_a and the current I_b.



The POG model of the given electric circuit is the following:



Let $\mathbf{x} = \begin{bmatrix} I_1 & V_2 & I_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \qquad\qquad\qquad\qquad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \qquad\qquad\qquad\qquad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \qquad\qquad \\ \\ \\ \\ \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \qquad\qquad\qquad\qquad \\ \\ \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} \qquad\qquad \\ \\ \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

20. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Mechanical Rotational.* Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols / Names	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$ q_1 v_1				
$\mathcal{D}_2$ q_2 v_2				
$\mathcal{R}$				

21. Given the following nonlinear system, continuous-time and autonomous:

$$\begin{cases} \dot{x}_1 &= -x_1^3 - x_2^4 \\ \dot{x}_2 &= (x_1 + \alpha)x_2 - x_2^3 \end{cases}$$

It is easy to verify that the point $(x_1,\ x_2) = (0,\ 0)$ is an equilibrium point for the system.

a) Linearize the system in the neighborhood of point $(x_1,\ x_2) = (0,\ 0)$ computing the matrix $\mathbf{A}$ of the corresponding linearized system:

b) Study, for varying parameter α, the stability of the nonlinear system in the neighborhood of point $(x_1,\ x_2) = (0,\ 0)$ using the reduced Lyapunov criterion:

c) For $\alpha = 0$, study the stability of the nonlinear system in the vicinity of the origin using the "direct" Lyapunov criterion and the following positive definite function: $V(\mathbf{x}) = x_1^2 + \frac{1}{2}x_2^4$.

22. How is it possible to compute the state transition matrix $\mathbf{A}^k$ of a discrete time-invariant linear system using the $\mathcal{Z}$ transfrom?

$$\mathbf{A}^k =$$