System and Control Theory
Test of January 25, 2011
Questions and Exercises

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1. Write the general solution of the linear time-variant differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$ being $\mathbf{x}(t_0)$ the state at time t_0:

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau$$

2. Write the explicit form of the *transition matrix* $\mathbf{\Phi}(k, h)$ of a discrete time-variant linear system $\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k)$:

$$\mathbf{\Phi}(k, h) = \begin{cases} \mathbf{A}(k-1)\dots\mathbf{A}(h+1)\mathbf{A}(h) & \text{if } k > h \\ \mathbf{I} \text{ (Identity matrix)} & \text{if } k = h \end{cases}$$

3. Write the explicit solution of the difference equation $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ being $\mathbf{x}(h)$ the state at time h.

$$\mathbf{x}(k) = \mathbf{A}^{k-h}\mathbf{x}(h) + \sum_{j=h}^{k-1} \mathbf{A}^{k-j-1}\mathbf{B}\mathbf{u}(j)$$

4. Compute the reachability matrix $\mathcal{R}^+$ and the observability matrix $\mathcal{O}^-$ of the following system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \qquad \mathcal{R}^+ = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathcal{O}^- = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix},$$

5. Given a linear POG dynamic system $\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ where $\mathbf{L}$ is a symmetric positive definite matrix. The transfer matrix $\mathbf{H}(s)$ which links the input vector $\mathbf{U}(s)$ to the vector d'uscita $\mathbf{Y}(s)$ can be expressed as follows:

   ○ $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$
   ⊗ $\mathbf{H}(s) = \mathbf{C}(s\mathbf{L} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$
   ○ $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{L}^{-1}\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$
   ⊗ $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{L}^{-1}\mathbf{A})^{-1}\mathbf{L}^{-1}\mathbf{B} + \mathbf{D}$

6. A matrix $\mathbf{A}$ of dimension n is diagonalizable

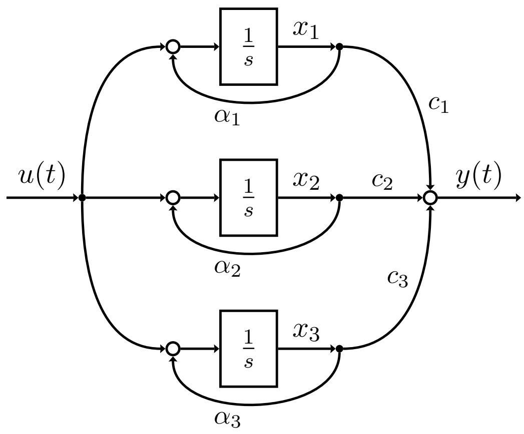
   ⊗ if it has n real distinct eigenvalues;
   ⊗ if it has n linearly independent eigenvectors;
   ⊗ if all the Jordan miniblocks have dimension equal to 1;
   ○ if all the eigenvalues have multiplicity greater than 1;

7. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ that move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the structure of the solution $\mathbf{u}$ which minimizes the Euclidean norm $||\mathbf{u}||$:

$$\mathbf{u} = (\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}[\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0)]$$

8. Draw the block scheme of the following continuous-time system in the Jordan canonical form where $\mathbf{x} = \left[\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}\right]^{\mathrm{T}}$.

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \\ \\ y(t) = \left[\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}\right] \mathbf{x}(t) \end{cases}$$



9. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^{\mathrm{T}}$, the free evolution of the following continuous-time autonomous system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \mathbf{x}(t) \qquad \mathbf{x}(k) = \begin{bmatrix} e^{2t} & t\,e^{2t} & \frac{t^2}{2}\,e^{2t} & 0 \\ 0 & e^{2t} & t\,e^{2t} & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

10. Given the transfer function $G(s)$, write the structure of corresponding dynamic system in the reachability canonical form denoting with $u(t)$ the input and with $y(t)$ the output:

$$G(s) = 2 + \frac{3s^3 + 6s^2 + 2s + 4}{s^4 + 2s^3 + 5s^2 + 3s + 7}$$

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7 & -3 & -5 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ \\ y(t) = \left[\begin{array}{cccc} 4 & 2 & 6 & 3 \end{array}\right] \mathbf{x}(t) + \left[\begin{array}{c} 2 \end{array}\right] u(t) \end{cases}$$

11. Given a SISO linear system of the forth order ($n = 4$), completely observable, characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$.
a) Write the structure of the matrices $\mathbf{A}_o$, $\mathbf{b}_o$ and $\mathbf{c}_o$ of the corresponding observability canonical form. Let $p(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ the characteristic polynomial of matrix $\mathbf{A}$.

$$\mathbf{A}_o = \begin{bmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{bmatrix}, \qquad \mathbf{b}_o = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}, \qquad \mathbf{c}_o = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right]$$

b) Moreover, write the structure of matrix $\mathbf{P}$ which, together with the space transformation $\mathbf{x} = \mathbf{P}\mathbf{x}_o$, brings the system in the observability canonical form.

$$\mathbf{P} = [(\mathcal{O}_c^-)^{-1}\mathcal{O}^-]^{-1} = \left(\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 1 \\ \alpha_2 & \alpha_3 & 1 & 0 \\ \alpha_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \mathbf{cA}^2 \\ \mathbf{cA}^3 \end{bmatrix} \right)^{-1}$$

12. Given a continuous-time SISO lieanr system: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t)$, $y(t) = \mathbf{c}\,\mathbf{x}(t)$ with $\mathbf{A} \in \mathbf{R}^{n\times n}$, $\mathbf{b} \in \mathbf{R}^{n\times 1}$ and $\mathbf{c} \in \mathbf{R}^{1\times n}$. Write the number N of non-constant parameters a_{ij}, b_i and c_j which characterize each canonical form of the given system:

$$N = 2\,n$$

13. Write the *Heymann Lemma*:
If $(\mathbf{A},\,\mathbf{B})$ *is reachable and if* $\mathbf{b}_i$ *is a not zero column of* $\mathbf{B}$*, then it exists a matrix* $\mathbf{M}_i \in \mathcal{R}^{m\times n}$ *such that* $(\mathbf{A}+\mathbf{B}\mathbf{M}_i,\,\mathbf{b}_i)$ *is reachable.*

14. Given the continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, write the structure of:
a) a *full order closed loop* state estimator:

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A}+\mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t)$$

b) the time evolution of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ obtained starting from the initial condition $\mathbf{e}(0)$:

$$\mathbf{e}(t) = e^{(\mathbf{A}+\mathbf{L}\mathbf{C})t}\,\mathbf{e}(0)$$

15. Write the structure of the matrix $\mathbf{P}^{-1}$ of the state space transformation $\mathbf{x} = \mathbf{P}\overline{\mathbf{x}}$ which brings a not-observable system in the standard observability form:

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} \qquad \text{where} \qquad \mathrm{Im}\mathbf{P}_1^{\mathrm{T}} = \mathrm{Im}(\mathcal{O}^-)^{\mathrm{T}} \text{ and } \mathbf{P}_2 \text{ makes non singular the matrix } \mathbf{P}^{-1}.$$

Moreover, write the block structure of the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$:

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{1,1} & 0 \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix} \qquad \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & 0 \end{bmatrix}$$

16. Relatively to the linear discrete system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k)+\mathbf{B}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)$, a necessary and sufficient condition for the complete "constructability" of the system:

$$\mathcal{E}^- = \ker \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} \subseteq \ker \mathbf{A}^n$$

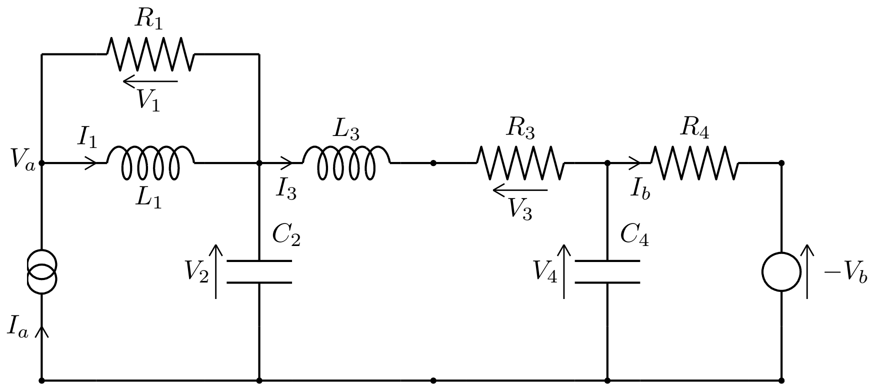
17. Write the direct Lyapunov stability criterion for continuous-time systems.

Consider the nonlinear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\,\mathbf{u}_0)$ *and let* $\mathbf{x}_0$ *be an equilibrium point corresponding to the constant input* $\mathbf{u}_0$*.*
1) If in a neighborhood W *of* $\mathbf{x}_0$ *it exists a function* $V(\mathbf{x}) : W \to \mathcal{R}$ positive definite *with continuous first time-derivatives and if* $\dot{V}(\mathbf{x})$ *is* negative semidefinite*, then the point* $\mathbf{x}_0$ *is* stable *for the nonlinear system .*
2) Moreover, if $\dot{V}(\mathbf{x})$ *is* negative definite*, then the the point* $\mathbf{x}_0$ *is* asymptotically *stable.*

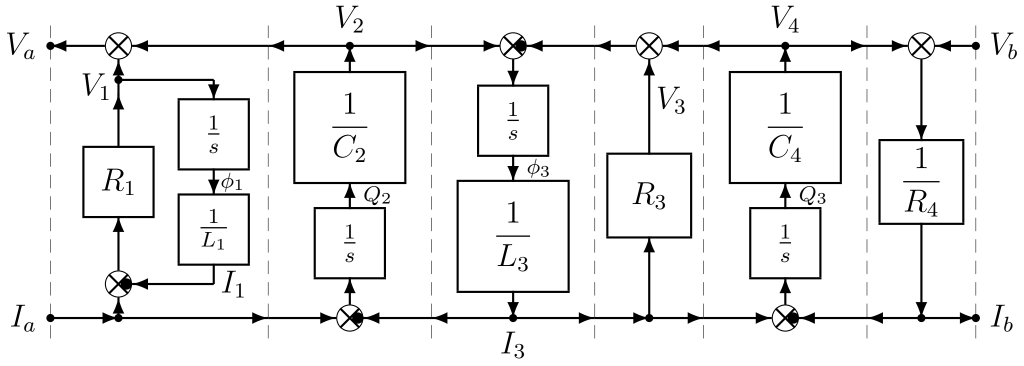
18. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$:

$$\mathbf{F} = e^{\mathbf{A}T}, \qquad \mathbf{G} = \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\,d\sigma, \qquad \mathbf{H} = \mathbf{C}$$

19. Consider the following electric circuit composed by the inductances L_1, L_3, the capacities C_2, C_3 and the resistances R_1, R_3 and R_4. Two inputs act on the system: the current I_a and the voltage V_b. The outputs of the system are: the voltage V_a and the current I_b.



The POG model of the given electric circuit is the following:



Let $\mathbf{x} = \begin{bmatrix} I_1 & V_2 & I_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector, $\mathbf{u} = \begin{bmatrix} I_a & V_b \end{bmatrix}^{\mathrm{T}}$ the input vector and $\mathbf{y} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ the output vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\mathbf{x} + \overline{\mathbf{B}}\mathbf{u}$ and $\mathbf{y} = \overline{\mathbf{C}}\mathbf{x} + \overline{\mathbf{D}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{V}_2 \\ \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -R_3 & -1 \\ 0 & 0 & 1 & -\frac{1}{R_4} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ V_2 \\ I_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} R_1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{R_4} \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -R_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R_4} \end{bmatrix}}_{\overline{\mathbf{C}}} \mathbf{x} + \underbrace{\begin{bmatrix} R_1 & 0 \\ 0 & \frac{1}{R_4} \end{bmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{bmatrix} I_a \\ V_b \end{bmatrix}}_{\mathbf{u}}$$

20. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Mechanical Rotational*. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which

characterize the physical elements:

	Symbols		Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$	J	Inertia			
q_1	P	Ang. Momentum	$P = \Phi_J(\omega)$	$P = J\,\omega$	$\dfrac{dP}{dt} = \tau$
v_1	ω	Ang. Velocity			
$\mathcal{D}_2$	E	Tors. Elasticity			
q_2	θ	Ang. Displacement	$\theta = \Phi_E(\tau)$	$\theta = E\,\tau$	$\dfrac{d\theta}{dt} = \omega$
v_2	τ	Torque			
$\mathcal{R}$	b	Friction	$\tau = \Phi_b(\omega)$	$\tau = b\,\omega$	

21. Given the following nonlinear system, continuous-time and autonomous:

$$\begin{cases} \dot{x}_1 &= -x_1^3 - x_2^4 \\ \dot{x}_2 &= (x_1 + \alpha)x_2 - x_2^3 \end{cases}$$

It is easy to verify that the point $(x_1,\, x_2) = (0,\, 0)$ is an equilibrium point for the system.

a) Linearize the system in the neighborhood of point $(x_1,\, x_2) = (0,\, 0)$ computing the matrix $\mathbf{A}$ of the corresponding linearized system:

The matrix $\mathbf{A}$ of the linearized system has the following structure:

$$\mathbf{A} = \left[\begin{array}{cc} -3x_1^2 & -4x_2^3 \\ x_2 & \alpha + x_1 - 3x_2^2 \end{array}\right]_{(x_1=0,\; x_2=0)} = \left[\begin{array}{cc} 0 & 0 \\ 0 & \alpha \end{array}\right]$$

b) Study, for varying parameter α, the stability of the nonlinear system in the neighborhood of point $(x_1,\ x_2) = (0,\ 0)$ using the reduced Lyapunov criterion:

The characteristic polynomial of matrix $\mathbf{A}$ is the following:

$$\Delta_{\mathbf{A}}(s) = s(s - \alpha) = 0$$

Using the reduced Lyapunov criterion it can be stated that: 1) for $\alpha > 0$ the nonlinear system is unstable in the vicinity of the origin; 2) for $\alpha \leq 0$ the criterion cannot be used.

c) For $\alpha = 0$, study the stability of the nonlinear system in the vicinity of the origin using the "direct" Lyapunov criterion and the following positive definite function: $V(\mathbf{x}) = x_1^2 + \frac{1}{2}x_2^4$.

Computing the time derivative of function $V(\mathbf{x})$ along the system's trajectories one obtains:

$$\dot{V} = 2x_1(-x_1^3 - x_2^4) + 2x_2^3(x_1 x_2 - x_2^3) = -2x_1^4 - 2x_2^6 < 0$$

Applying the "direct" Lyapunov criterion it can be stated that in the neighborhood of the origin $\mathbf{x}_0 = 0$ the nonlinear system is asymptotically stable.

22. How is it possible to compute the state transition matrix $\mathbf{A}^k$ of a discrete time-invariant linear system using the $\mathcal{Z}$ transfrom?

$$\mathbf{A}^k = \mathcal{Z}^{-1}[z(z\mathbf{I} - \mathbf{A})^{-1}]$$