System and Control Theory
Test of Decembre 23, 2010
Questions and Exercises

Name:	
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Signature:	

1. Write the number and the type of parameters that characterize the *state transfer function* of a dynamic continuous-time system:

$$x(t) = \psi(\qquad\qquad\qquad)$$

2. Write the closed form solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$:

$$\mathbf{x}(t) =$$

3. The *geometric* multiplicity of an eigenvalue λ of a matrix $\mathbf{A}$

   - ◯ is the dimension of the Jordan block $\mathbf{J}_\lambda$ associated to the eigenvalue λ;
   - ◯ is equal to the number of miniblocks of Jordan associated to the eigenvalue λ;
   - ◯ is the multiplicity degree of λ in the characteristic polynomial of matrix $\mathbf{A}$;
   - ◯ is the number of eigenvectors linearly independent associated to the eigenvalue λ;

4. Write the transfer matrices $\mathbf{H}(z)$ of a discrete-time linear system as a function of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ that characterize the linear system:

$$\mathbf{H}(z) =$$

5. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ that move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the structure of the solution $\mathbf{u}$ which minimizes the Euclidean norm $||\mathbf{u}||$:

$$\mathbf{u} =$$

6. Draw the block scheme of the following continuous-time system in the observability canonical form where $\mathbf{x}_o = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$.

$$\begin{cases} \dot{\mathbf{x}}_o(t) = \begin{bmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_o(t) + d_0\, u(t) \end{cases}$$

7. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^{\mathrm{T}}$, the free evolution of the following discrete-time autonomous system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \mathbf{x}(k) \qquad \mathbf{x}(k) = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

8. Given a SISO linear system of the forth order ($n = 4$), completely reachable, characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$.
a) Write the structure of the matrices $\mathbf{A}_c$, $\mathbf{b}_c$ and $\mathbf{c}_c$ of the corresponding controllability canonical form. Let $p(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ the characteristic polynomial of matrix $\mathbf{A}$.

$$\mathbf{A}_c = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}, \qquad \mathbf{b}_c = \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix}, \qquad \mathbf{c}_c = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}$$

b) Moreover, write the structure of matrix $\mathbf{T}$ which, together with the space transformation $\mathbf{x} = \mathbf{T}\mathbf{x}_c$, brings the system in the controllability canonical form.

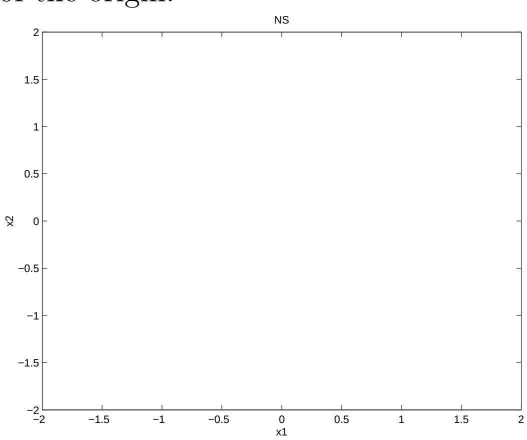
$$\mathbf{T} =$$

c) Write the transfer function $G(s)$ corresponding to the controllability canonical form reported above:

$$G(s) = \frac{Y(s)}{U(s)} =$$

9. Considered a continuous-time dynamic system of the second order characterized by two real eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$, answer to the following questions and draw the qualitative behavior of the state trajectories in the vicinity of the origin:

- ◯ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are real and different.
- ◯ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are complex conjugate.
- ◯ for $t \to \infty$ all the trajectories tend to flatten on the eigenvector $\mathbf{v}_1$.
- ◯ for $t \to \infty$ all the trajectories tend to flatten on the eigenvector $\mathbf{v}_2$.



Which name is typically used for denoting the type of trajectories shown above:

◯ Node? ◯ Focus? ◯ Saddle? ◯ Degenerate? ◯ Stable? ◯ Unstable?

10. a) Write the explicit form of the Ackermann formula which provides the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of a feedback system:

$$\mathbf{k}^{\mathrm{T}} =$$

b) Write the structure of the desired polynomial $p(\lambda)$ and the matrix $p(\mathbf{A})$ when $n = 4$, two systems's eigenvalues must be located in $\lambda = -2$ and the other two systems's eigenvalues must be located in $\lambda = -1$.

$$p(\lambda) = \qquad\qquad p(\mathbf{A}) =$$

11. Given the discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, write the structure of:

a) *open loop* state estimator:

$$\hat{\mathbf{x}}(k+1) =$$

b) a *full order closed loop* state estimator:

$$\hat{\mathbf{x}}(k+1) =$$

c) the time evolution of the estimation errors $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ in the two previous cases a) and b) starting from the initial condition $\mathbf{e}(0)$:

$$\mathbf{e}(k) = \qquad\qquad \mathbf{e}(k) =$$

12. Given a continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$:

a) it is possible to use an asymptotic *open loop* state observer if and only if:

...

b) it is possible to use an asymptotic *full order closed loop* state observer if and only if:

...

c) it is possible to use an asymptotic *a reduced order closed loop* state observer if and only if:

...

13. Write the structure of the dual system $\mathcal{S}_D$ corresponding to a given system $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$:

$$\mathcal{S}_D = (\quad , \quad , \quad , \quad)$$

14. Write the structure of matrix $\mathbf{T}$ of the state space transformation $\mathbf{x} = \mathbf{T}\overline{\mathbf{x}}$ which brings a not completely reachable system in the reachability standard form:

$$\mathbf{T} =$$

Moreover, write the block structure of the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$.

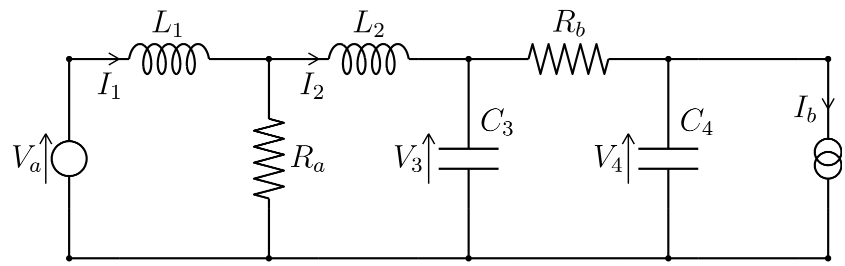
$$\overline{\mathbf{A}} = \begin{bmatrix} \quad \end{bmatrix}, \qquad \overline{\mathbf{B}} = \begin{bmatrix} \quad \end{bmatrix}, \qquad \overline{\mathbf{C}} = \begin{bmatrix} \quad \end{bmatrix}$$

15. Write the necessary and sufficient condition which guarantees the controllability in k steps of the discrete linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$:

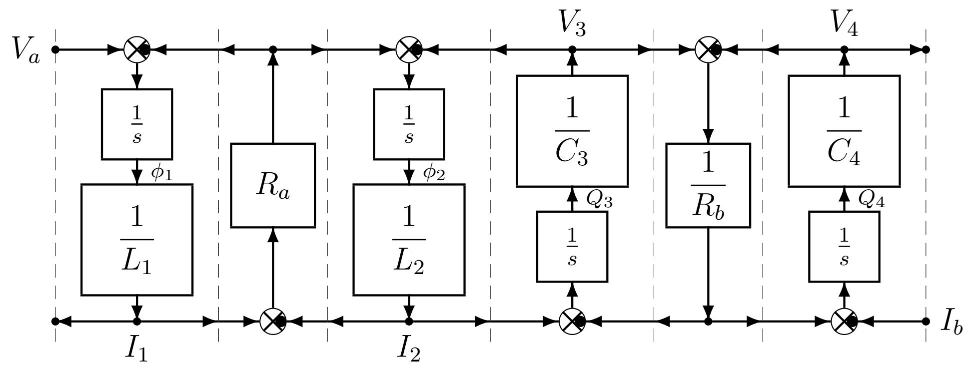
16. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$:

$\mathbf{F} =$ $\qquad\qquad$ $\mathbf{G} =$ $\qquad\qquad$ $\mathbf{H} =$

17. Consider the following electric circuit composed by the inductances L_1, L_2, the capacities C_3, C_3, the resistances R_a, R_b, the input voltage V_a and the output current I_b:



The POG model of the given electric circuit is the following:



Let $\mathbf{x} = \begin{bmatrix} I_1 & I_2 & V_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector (composed by the output power variables of the dynamic elements) and let $\mathbf{u} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ be the input vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\dot{\mathbf{x}} + \overline{\mathbf{B}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

18. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Hydraulic*. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize the physical elements:

	Symbols / Names	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$ q_1 v_1				
$\mathcal{D}_2$ q_2 v_2				
$\mathcal{R}$				

19. Given the following nonlinear system, continuous-time and autonomous:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_2 - x_1 - x_2^3 \end{cases}$$

It is easy to verify that the point $(x_1,\ x_2) = (0,\ 0)$ is an equilibrium point for the system.

a) Linearize the system in the neighborhood of point $(x_1,\ x_2) = (0,\ 0)$ computing the matrix $\mathbf{A}$ of the corresponding linearized system:

b) Study, for varying parameter α, the stability of the nonlinear system in the neighborhood of point $(x_1,\ x_2) = (0,\ 0)$ using the reduced Lyapunov criterion:

c) For $\alpha = 0$, study the stability of the nonlinear system in the vicinity of the origin using the "direct" Lyapunov criterion and the following positive definite function: $V(\mathbf{x}) = x_1^2 + x_2^2$. Eventually, use the criterion of La Salle - Krasowskii.