System and Control Theory
Test of Decembre 23, 2010
Questions and Exercises

Name:	
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1. Write the number and the type of parameters that characterize the *state transfer function* of a dynamic continuous-time system:

$$x(t) = \psi(t,\, t_0, x(t_0), u(\cdot))$$

2. Write the closed form solution of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ starting from the initial condition $\mathbf{x}(t_0)$:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

3. The *geometric* multiplicity of an eigenvalue λ of a matrix $\mathbf{A}$

   ○ is the dimension of the Jordan block $\mathbf{J}_\lambda$ associated to the eigenvalue λ;
   ⊗ is equal to the number of miniblocks of Jordan associated to the eigenvalue λ;
   ○ is the multiplicity degree of λ in the characteristic polynomial of matrix $\mathbf{A}$;
   ⊗ is the number of eigenvectors linearly independent associated to the eigenvalue λ;

4. Write the transfer matrices $\mathbf{H}(z)$ of a discrete-time linear system as a function of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ that characterize the linear system:

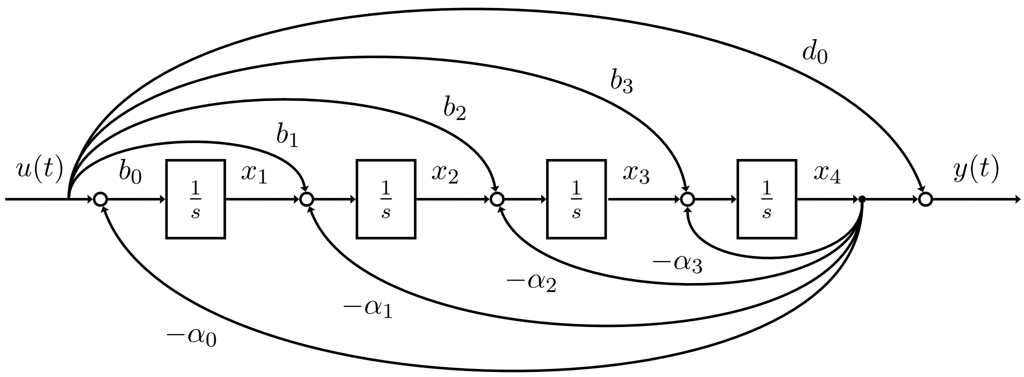
$$\mathbf{H}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

5. Consider the point-to-point control problem for a discrete-time linear system. Among the infinite solutions $\mathbf{u}$ that move the system from the initial state $\mathbf{x}(0)$ to the final state $\mathbf{x}(k)$ in the time interval $[0,\ k]$ write the structure of the solution $\mathbf{u}$ which minimizes the Euclidean norm $||\mathbf{u}||$:

$$\mathbf{u} = (\mathcal{R}_k^+)^T[\mathcal{R}_k^+(\mathcal{R}_k^+)^T]^{-1}[\mathbf{x}(k) - \mathbf{A}^k\mathbf{x}(0)]$$

6. Draw the block scheme of the following continuous-time system in the observability canonical form where $\mathbf{x}_o = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}}$.

$$\left\{\begin{array}{rcl} \dot{\mathbf{x}}_o(t) & = & \begin{bmatrix} 0 & 0 & 0 & -\alpha_0 \\ 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & -\alpha_2 \\ 0 & 0 & 1 & -\alpha_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_o(t) \;+\; d_0\, u(t) \end{array}\right.$$



7. Compute, as function of the initial condition $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^{\mathrm{T}}$, the free evolution of the following discrete-time autonomous system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \mathbf{x}(k) \qquad \mathbf{x}(k) = \begin{bmatrix} 2^k & k\,2^{k-1} & 0 & 0 \\ 0 & 2^k & k\,0 & 0 \\ 0 & 0 & (-2)^k & k\,(-2)^{k-1} \\ 0 & 0 & 0 & (-2)^k \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix}$$

8. Given a SISO linear system of the forth order ($n = 4$), completely reachable, characterized by matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$.
a) Write the structure of the matrices $\mathbf{A}_c$, $\mathbf{b}_c$ and $\mathbf{c}_c$ of the corresponding controllability canonical form. Let $p(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ the characteristic polynomial of matrix $\mathbf{A}$.

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix}, \qquad \mathbf{b}_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{c}_c = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

b) Moreover, write the structure of matrix $\mathbf{T}$ which, together with the space transformation $\mathbf{x} = \mathbf{T}\mathbf{x}_c$, brings the system in the controllability canonical form.

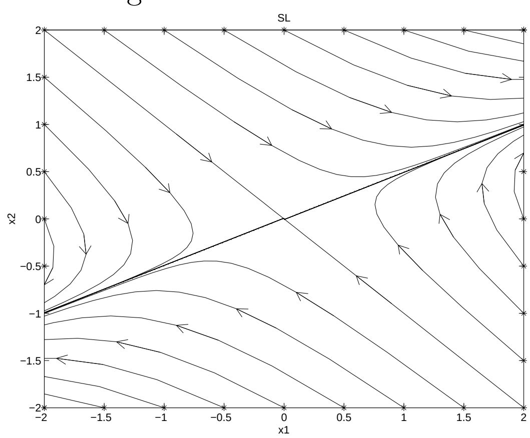
$$\mathbf{T} = \mathcal{R}^+(\mathcal{R}_c^+)^{-1} = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 1 \\ \alpha_2 & \alpha_3 & 1 & 0 \\ \alpha_3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

c) Write the transfer function $G(s)$ corresponding to the controllability canonical form reported above:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}$$

9. Considered a continuous-time dynamic system of the second order characterized by two real eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$, answer to the following questions and draw the qualitative behavior of the state trajectories in the vicinity of the origin:

- ⊗ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are real and different.
- ○ the system's eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are complex conjugate.
- ⊗ for $t \to \infty$ all the trajectories tend to flatten on the eigenvector $\mathbf{v}_1$.
- ○ for $t \to \infty$ all the trajectories tend to flatten on the eigenvector $\mathbf{v}_2$.



Which name is typically used for denoting the type of trajectories shown above:

○ Node? ○ Focus? ⊗ Saddle? ○ Degenerate? ○ Stable? ⊗ Unstable?

10. a) Write the explicit form of the Ackermann formula which provides the vector $\mathbf{k}^{\mathrm{T}}$ allowing the free positioning of the eigenvalues of a feedback system:

$$\mathbf{k}^{\mathrm{T}} = -\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} (\mathcal{R}^+)^{-1} p(\mathbf{A})$$

b) Write the structure of the desired polynomial $p(\lambda)$ and the matrix $p(\mathbf{A})$ when $n = 4$, two systems's eigenvalues must be located in $\lambda = -2$ and the other two systems's eigenvalues must be located in $\lambda = -1$.

$$p(\lambda) = (\lambda + 2)^2(\lambda + 1)^2, \qquad\qquad p(\mathbf{A}) = (\mathbf{A} + 2\mathbf{I})^2(\mathbf{A} + \mathbf{I})^2$$

11. Given the discrete-time linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, write the structure of:

a) *open loop* state estimator:

$$\hat{\mathbf{x}}(k+1) = \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k)$$

b) a *full order closed loop* state estimator:

$$\hat{\mathbf{x}}(k+1) = (\mathbf{A} + \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) - \mathbf{L}\mathbf{y}(k)$$

c) the time evolution of the estimation errors $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ in the two previous cases a) and b) starting from the initial condition $\mathbf{e}(0)$:

$$\mathbf{e}(k) = \mathbf{A}^k\mathbf{e}(0), \qquad\qquad \mathbf{e}(k) = (\mathbf{A} + \mathbf{L}\mathbf{C})^k\mathbf{e}(0)$$

12. Given a continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$:

a) it is possible to use an asymptotic *open loop* state observer if and only if:
the system is asymptotically stable

b) it is possible to use an asymptotic *full order closed loop* state observer if and only if:
the not-observable part of the system is asymptotically stable

c) it is possible to use an asymptotic *a reduced order closed loop* state observer if and only if:
the not-observable part of the system is asymptotically stable

13. Write the structure of the dual system $\mathcal{S}_D$ corresponding to a given system $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$:

$$\mathcal{S}_D = (\mathbf{A}^{\mathrm{T}},\ \mathbf{C}^{\mathrm{T}},\ \mathbf{B}^{\mathrm{T}},\ \mathbf{D}^{\mathrm{T}})$$

14. Write the structure of matrix $\mathbf{T}$ of the state space transformation $\mathbf{x} = \mathbf{T}\overline{\mathbf{x}}$ which brings a not completely reachable system in the reachability standard form:

$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \end{bmatrix}$ where $\mathrm{Im}\mathbf{T}_1 = \mathrm{Im}\mathcal{R}^+ = \mathcal{X}^+$ and $\mathbf{T}_2$ makes non singular the matrix $\mathbf{T}$.

Moreover, write the block structure of the matrices $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$ and $\overline{\mathbf{C}}$.

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ 0 & \mathbf{A}_{2,2} \end{bmatrix} \quad \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}$$

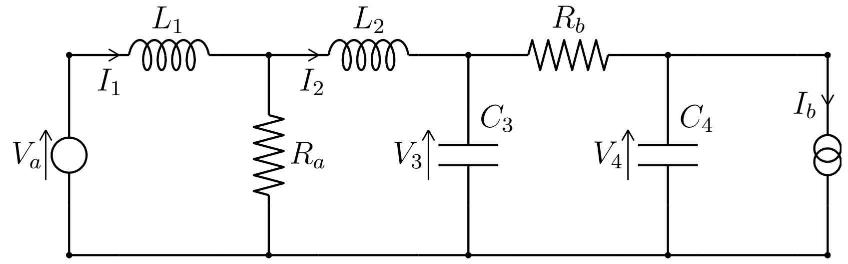
15. Write the necessary and sufficient condition which guarantees the controllability in k steps of the discrete linear system $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$:

$$\mathrm{Im}\mathbf{A}^k \subseteq \mathrm{Im}[\mathbf{B}, \mathbf{A}\mathbf{B}, \ldots, \mathbf{A}^{k-1}\mathbf{B}] = \mathcal{X}^+(k)$$

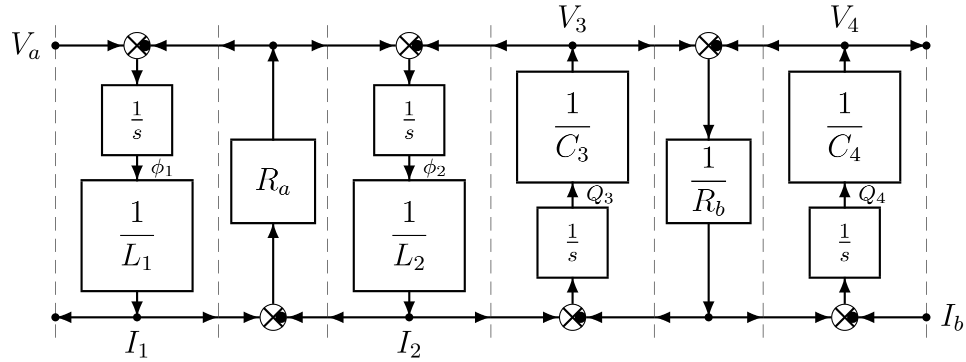
16. Given the following continuous-time linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. Write the expression of the matrices $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ that characterize the corresponding sampled system $\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$, $\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k)$:

$$\mathbf{F} = e^{\mathbf{A}T}, \qquad \mathbf{G} = \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\, d\sigma, \qquad \mathbf{H} = \mathbf{C}$$

17. Consider the following electric circuit composed by the inductances L_1, L_2, the capacities C_3, C_3, the resistances R_a, R_b, the input voltage V_a and the output current I_b:



The POG model of the given electric circuit is the following:



Let $\mathbf{x} = \begin{bmatrix} I_1 & I_2 & V_3 & V_4 \end{bmatrix}^{\mathrm{T}}$ be the state vector (composed by the output power variables of the dynamic elements) and let $\mathbf{u} = \begin{bmatrix} V_a & I_b \end{bmatrix}^{\mathrm{T}}$ be the input vector. Write the corresponding dynamic system $\overline{\mathbf{L}}\dot{\mathbf{x}} = \overline{\mathbf{A}}\dot{\mathbf{x}} + \overline{\mathbf{B}}\mathbf{u}$ in the state space:

$$\underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_a & R_a & 0 & 0 \\ R_a & -R_a & -1 & 0 \\ 0 & 1 & -\frac{1}{R_b} & \frac{1}{R_b} \\ 0 & 0 & \frac{1}{R_b} & -\frac{1}{R_b} \end{bmatrix}}_{\overline{\mathbf{A}}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ V_3 \\ V_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\overline{\mathbf{B}}} \underbrace{\begin{bmatrix} V_a \\ I_b \end{bmatrix}}_{\mathbf{u}}$$

18. Write, within the following table, the symbols and the names of the energy variables and the power variables that characterize the Energetic Domain: *Hydraulic*. Moreover, write the constitutive relation (both linear and nonlinear) and the differential equation which characterize

the physical elements:

		Symbols	Constitutive Rel.	Linear Case	Differential Eq.
$\mathcal{D}_1$	C_I	Hydr. Capacity			
q_1	V	Volume	$V = \Phi_C(P)$	$V = C_I\, P$	$\dfrac{dV}{dt} = Q$
v_1	P	Pressure			
$\mathcal{D}_2$	L_I	Hydr. Inductance			
q_2	ϕ_I	Hydr. Flux	$\phi_I = \Phi_L(Q)$	$\phi_I = L_I\, Q$	$\dfrac{d\,\phi_I}{dt} = P$
v_2	Q	Volume flow rate			
$\mathcal{R}$	R	Hydr. Resistance	$P = \Phi_R(Q)$	$P = R_I\, Q$	

19. Given the following nonlinear system, continuous-time and autonomous:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_2 - x_1 - x_2^3 \end{cases}$$

It is easy to verify that the point $(x_1,\, x_2) = (0,\, 0)$ is an equilibrium point for the system.

a) Linearize the system in the neighborhood of point $(x_1,\, x_2) = (0,\, 0)$ computing the matrix $\mathbf{A}$ of the corresponding linearized system:

The matrix $\mathbf{A}$ of the linearized system has the following structure:

$$\mathbf{A} = \left[\begin{array}{cc} 0 & 1 \\ -1 & \alpha - 3x_2^2 \end{array}\right]_{(x_1=0,\; x_2=0)} = \left[\begin{array}{cc} 0 & 1 \\ -1 & \alpha \end{array}\right]$$

b) Study, for varying parameter α, the stability of the nonlinear system in the neighborhood of point $(x_1,\; x_2) = (0,\; 0)$ using the reduced Lyapunov criterion:

The characteristic polynomial of matrix $\mathbf{A}$ is the following:

$$\Delta_{\mathbf{A}}(s) = s^2 - \alpha\, s + 1 = 0$$

Using the reduced Lyapunov criterion it can be stated that: 1) for $\alpha > 0$ the nonlinear system is unstable in the vicinity of the origin; 2) for $\alpha < 0$ the system is asymptotically stable in the vicinity of the origin; 3) for $\alpha = 0$ the criterion cannot be used.

c) For $\alpha = 0$, study the stability of the nonlinear system in the vicinity of the origin using the “direct” Lyapunov criterion and the following positive definite function: $V(\mathbf{x}) = x_1^2 + x_2^2$. Eventually, use the criterion of La Salle - Krasowskii.

Computing the time derivative of function $V(\mathbf{x})$ along the system’s trajectories one obtains:

$$\dot{V} = 2x_1x_2 + 2x_2(-x_1 - x_2^3) = -2x_2^4 \leq 0$$

Applying the “direct” Lyapunov criterion it can be stated that in the neighborhood of the origin $\mathbf{x}_0 = 0$ the nonlinear system is stable. The set $\mathcal{N} = \{(x_1, 0), x_1 \in R\}$ of all the points that nullify the function $\dot{V}$ does not contain perturbed trajectories of the system and therefore, using the La Salle - Krasowskii criterion, it can be stated that for $\alpha = 0$ the nonlinear system is asymptotically stable in the vicinity of the origin.