Laplace transform for continuous linear systems.

- Applying the Laplace transform to the *state* and *output* vectors of a linear system:

$$\mathcal{L}\left(\begin{array}{l}\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t)\end{array}\right)=\left(\begin{array}{l}s\mathbf{x}(s)-\mathbf{x}_0=\mathbf{A}\mathbf{x}(s)+\mathbf{B}\mathbf{u}(s)\\ \mathbf{y}(s)=\mathbf{C}\mathbf{x}(s)+\mathbf{D}\mathbf{u}(s)\end{array}\right)$$

one obtains the following relations:

$$\begin{cases}\mathbf{x}(s)=(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0+(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s)\\ \mathbf{y}(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0+[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]\mathbf{u}(s)\end{cases}$$

- When the input is zero, $\mathbf{u}(t)=0$, one obtains the <u>*free evolution*</u>:

$$\begin{cases}\mathbf{x}(s)=(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0\\ \mathbf{y}(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0\end{cases}\quad\leftrightarrow\quad\begin{cases}\mathbf{x}(t)=e^{\mathbf{A}t}\mathbf{x}_0\\ \mathbf{y}(t)=\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0\end{cases}$$

from which one obtains:

$$\boxed{e^{\mathbf{A}t}=\mathcal{L}^{-1}[(s\mathbf{I}-\mathbf{A})^{-1}]}$$

- When the initial state is zero, $\mathbf{x}_0=0$, one obtains the <u>*forced evolution*</u>:

$$\begin{cases}\mathbf{x}(s)=(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s)\\ \mathbf{y}(s)=[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]\mathbf{u}(s)\end{cases}\quad\leftrightarrow\quad\begin{cases}\mathbf{x}(t)=\displaystyle\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\\ \mathbf{y}(t)=\mathbf{C}\displaystyle\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau+\mathbf{D}\mathbf{u}(t)\end{cases}$$

Transfer matrix $\mathbf{H}(s)$: is the *fractional* matrix of dimension $(p\times m)$ defined as follows:

$$\mathbf{y}(s)=\mathbf{H}(s)\mathbf{u}(s)\qquad\rightarrow\qquad\boxed{\mathbf{H}(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}}$$

- Example in Matlab:

```
---------------------------------------------------------------------------
                                  +-                                  -+
A=[0 1; -2 -3];                   |     s - 2          2      1        |
B=[1 0; 1 1];                     |  --------------- + 1, ----- - ----- |
C=[0 1; 1 -1];                    |  (s + 1) (s + 2)      s + 2   s + 1 |
D=[1 0; 0 0];          -->        |                                    |
syms s                            |         6              2      3    |
Hs=C*inv(s*eye(2)-A)*B+D;         |    ---------------,  ----- - ----- |
pretty(simplify(Hs))              |    (s + 1) (s + 2)   s + 1   s + 2 |
                                  +-                                  -+
---------------------------------------------------------------------------
```

• **Transfer matrix $\mathbf{H}(s)$**:

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} = \mathbf{C}\,\frac{\mathrm{agg}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}\,\mathbf{B}+\mathbf{D}.$$

• The *inverse* of a nonsingular square matrix $\mathbf{M}$ of order n is defined as follows:

$$\mathbf{M}^{-1} = \frac{\mathrm{agg}\,\mathbf{M}}{\det\;\mathbf{M}}$$

The *"adjoined matrix"* $\mathrm{agg}\,\mathbf{M}$ is the *transpose* of the matrix of the algebraic complements $M_{i,j}$ of matrix $\mathbf{M}$. The algebraic complement $M_{i,j}$ is $(-1)^{i+j}$ times the determinant of $(n-1)\times(n-1)$ submatrix obtained from $\mathbf{M}$ eliminating the i-th row and the $j-$th column.

• $\det(s\mathbf{I}-\mathbf{A})$ is e polynomial of degree n.

• All the elements of the polynomial matrix $\mathrm{agg}(s\mathbf{I}-\mathbf{A})$ have degree less or equal to $n-1$.

• All the fractional functions present in matrix $\mathbf{H}(s)-\mathbf{D} = \mathbf{C}\;(s\mathbf{I}-\mathbf{A})^{-1}\;\mathbf{B}$ are strictly proper: $n > m$.

• Property. The transfer matrix $\mathbf{H}(s)$ does not change if one applies a state space transformation $\mathbf{x} = \mathbf{T}\,\overline{\mathbf{x}}$ to the considered system:

$$\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \qquad \overline{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}, \qquad \overline{\mathbf{C}} = \mathbf{C}\mathbf{T}, \qquad \overline{\mathbf{D}} = \mathbf{D}.$$

In fact, the following relations hold:

$$\begin{aligned}
\mathbf{H}(s) &= \overline{\mathbf{C}}(s\mathbf{I}-\overline{\mathbf{A}})^{-1}\overline{\mathbf{B}}+\overline{\mathbf{D}}\\
&= \mathbf{C}\mathbf{T}(s\mathbf{I}-\mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{-1}\mathbf{T}^{-1}\mathbf{B}+\mathbf{D}\\
&= \mathbf{C}\mathbf{T}\left[\mathbf{T}^{-1}(s\mathbf{I}-\mathbf{A})\mathbf{T}\right]^{-1}\mathbf{T}^{-1}\mathbf{B}+\mathbf{D}\\
&= \mathbf{C}\mathbf{T}[\mathbf{T}^{-1}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{T}]\mathbf{T}^{-1}\mathbf{B}+\mathbf{D}\\
&= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}
\end{aligned}$$

• Property. The static gain $\mathbf{H}_0$ is obtained from $\mathbf{H}(s)$ with $s=0$:

$$\mathbf{H}_0 = \mathbf{H}(s)|_{s=0} = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B}+\mathbf{D} \qquad \rightarrow \qquad \mathbf{y}_0 = \mathbf{H}_0\,\mathbf{u}_0.$$

If the input $\mathbf{u}_0$ is constant the steady state value of the output is $\mathbf{y}_0 = \mathbf{H}_0\,\mathbf{u}_0$.

Example. Compute the exponential matrix of the following matrix $\mathbf{A}$:

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] \qquad \text{dove} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The characteristic polynomial of matrix $\mathbf{A}$ is:

$$\det(s\mathbf{I}-\mathbf{A}) = \det\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} = s^2+3s+2 = (s+1)(s+2)$$

It follows that (in Matlab: `syms s; As=inv(s*eye(2)-A); pretty(As)`):

$$\mathbf{A}_s = (s\mathbf{I}-\mathbf{A})^{-1} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{(s+1)(s+2)} = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

The inverse Laplace transform of the elements of matrix $\mathbf{A}_s$ are:

$$\mathcal{L}^{-1}\left[\frac{s+3}{(s+1)(s+2)}\right] = \mathcal{L}^{-1}\left[\frac{2}{(s+1)} - \frac{1}{(s+2)}\right] = 2e^{-t} - e^{-2t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)} - \frac{1}{(s+2)}\right] = e^{-t} - e^{-2t}$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s+1)(s+2)}\right] = \mathcal{L}^{-1}\left[\frac{-1}{(s+1)} + \frac{2}{(s+2)}\right] = -e^{-t} + 2e^{-2t}$$

Substituting, one obtains that (in Matlab: `Eat=ilaplace(As); pretty(Eat)`):

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[\mathbf{A}_s] = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

- Example of direct computation of matrix $e^{\mathbf{A}t}$ in Matlab:

```
---------------------------------------------------------------------------------------
                                +-                                                  -+
A=[0 1; -2 -3];                 |   2 exp(-t) - exp(-2 t),    exp(-t) - exp(-2 t)    |
syms t;                -->      |                                                    |
Eat=expm(A*t);                  |  2 exp(-2 t) - 2 exp(-t), 2 exp(-2 t) - exp(-t)    |
pretty(Eat)                     +-                                                  -+
---------------------------------------------------------------------------------------
```

$\mathcal{Z}$ transform for discrete linear systems

• Applying the $\mathcal{Z}$ transform to the *state* and *output* functions of a time-invariant discrete linear system, one obtains that:

$$\mathcal{Z}\left(\left\{\begin{array}{l}\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+\mathbf{B}\mathbf{u}(k)\\ \mathbf{y}(k)=\mathbf{C}\mathbf{x}(k)+\mathbf{D}\mathbf{u}(k)\end{array}\right.\right)=\left\{\begin{array}{l}z(\mathbf{x}(z)-\mathbf{x}_0)=\mathbf{A}\mathbf{x}(z)+\mathbf{B}\mathbf{u}(z)\\ \mathbf{y}(z)=\mathbf{C}\mathbf{x}(z)+\mathbf{D}\mathbf{u}(z)\end{array}\right.$$

• Solving the system with respect to vector $\mathbf{x}(z)$ one obtains:

$$\left\{\begin{array}{l}\mathbf{x}(z)=(z\mathbf{I}-\mathbf{A})^{-1}z\mathbf{x}_0+(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{u}(z)\\ \mathbf{y}(z)=\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}z\mathbf{x}_0+[\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]\mathbf{u}(z)\end{array}\right.$$

• When the input is zero, $\mathbf{u}(k)=0$, one obtains the l'*free evolution*:

$$\left\{\begin{array}{l}\mathbf{x}(z)=(z\mathbf{I}-\mathbf{A})^{-1}z\mathbf{x}_0\\ \mathbf{y}(z)=\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}z\mathbf{x}_0\end{array}\right. \qquad \leftrightarrow \qquad \left\{\begin{array}{l}\mathbf{x}(k)=\mathbf{A}^k\mathbf{x}_0\\ \mathbf{y}(k)=\mathbf{C}\mathbf{A}^k\mathbf{x}_0\end{array}\right.$$

• The following relation holds:

$$\boxed{\mathbf{A}^k=\mathcal{Z}^{-1}[z(z\mathbf{I}-\mathbf{A})^{-1}]}$$

• When the initial state is zero, $\mathbf{x}_0=0$, one obtains the *forced evolution*:

$$\left\{\begin{array}{l}\mathbf{x}(z)=(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{u}(z).\\ \mathbf{y}(z)=\underbrace{[\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]}_{\mathbf{H}(z)}\mathbf{u}(z)\end{array}\right. \qquad \leftrightarrow \qquad \left\{\begin{array}{l}\mathbf{x}(k)=\displaystyle\sum_{j=0}^{k-1}\mathbf{A}^{(k-j-1)}\mathbf{B}\mathbf{u}(j)\\ \mathbf{y}(k)=\displaystyle\sum_{j=0}^{k-1}\mathbf{A}^{(k-j-1)}\mathbf{B}\mathbf{u}(j)+\mathbf{D}\mathbf{u}(k)\end{array}\right.$$

• **Transfer matrix $\mathbf{H}(z)$**: is the *fractional* matrix of dimension $(p \times m)$ defined as follows:

$$\mathbf{y}(z)=\mathbf{H}(z)\mathbf{u}(z) \qquad \rightarrow \qquad \boxed{\mathbf{H}(z)=\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}}$$

All the fractional functions of matrix $\mathbf{H}(z)$ have relative degree $r \geq 0$. If $\mathbf{D}=0$ all the functions of matrix $\mathbf{H}(z)$ are strictly proper: $r>0$.

• The static gain $\mathbf{H}_0$ of matrix $\mathbf{H}(z)$ is obtained from $\mathbf{H}(z)$ with $z=1$:

$$\mathbf{H}_0=\mathbf{H}(z)|_{z=1}=\mathbf{C}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} \qquad \rightarrow \qquad \mathbf{y}_0=\mathbf{H}_0\,\mathbf{u}_0.$$

<u>*Example*</u>. Compute the free evolution of the following autonomous discrete system:

$$\mathbf{x}(n+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{\mathbf{A}} \mathbf{x}(n)$$

starting from the initial condition $x(0) = \mathbf{x}_0$.

<u>*Solution*</u>. The solution of the problem is:

$$\mathbf{x}(n) = \mathbf{A}^n \mathbf{x}_0$$

Matrix $\mathbf{A}^n$ can be computed using the following relation:

$$\mathbf{A}^n = \mathcal{Z}^{-1}\left[(z\mathbf{I} - A)^{-1} z\right]$$

The characteristic polynomial and the eigenvalues of matrix $\mathbf{A}$ are: $p(\lambda) = \lambda^2 + 3\lambda + 2$, $\lambda_1 = -1$ and $\lambda_2 = -2$. From which it follows that:

$$\mathbf{A}^n = \mathcal{Z}^{-1}\left\{ \frac{\begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix} z}{(z+1)(z+2)} \right\} = \mathcal{Z}^{-1}\left\{ \begin{bmatrix} \frac{z+3}{(z+1)(z+2)} & \frac{1}{(z+1)(z+2)} \\ \frac{-2}{(z+1)(z+2)} & \frac{z}{(z+1)(z+2)} \end{bmatrix} z \right\}$$

$$= \mathcal{Z}^{-1}\left\{ \begin{bmatrix} \frac{2\,z}{(z+1)} - \frac{z}{(z+2)} & \frac{z}{(z+1)} - \frac{z}{(z+2)} \\ \frac{-2\,z}{(z+1)} + \frac{2\,z}{(z+2)} & \frac{-z}{(z+1)} + \frac{2\,z}{(z+2)} \end{bmatrix} \right\}$$

Using the inverse $\mathcal{Z}$ transform one obtains:

$$\mathbf{A}^n = \begin{bmatrix} 2(-1)^n - (-2)^n & (-1)^n - (-2)^n \\ -2(-1)^n + 2(-2)^n & -(-1)^n + 2(-2)^n \end{bmatrix}$$

- Example of direct computation of matrix $\mathbf{A}^k$ in Matlab:

```
---------------------------------------------------------------------------------------
                                   +-                                      -+
 syms z                            |          n        n         n        n |
 A=[0 1; -2 -3];                   |  2 (-1)  - (-2) ,    (-1)  - (-2)      |
 Ad=z*inv(z*eye(2)-A);   -->       |                                        |
 Ak=iztrans(Ad);                   |         n         n         n        n |
 pretty(Ak)                        | 2 (-2)  - 2 (-1) , 2 (-2)  - (-1)     |
                                   +-                                      -+
---------------------------------------------------------------------------------------
```