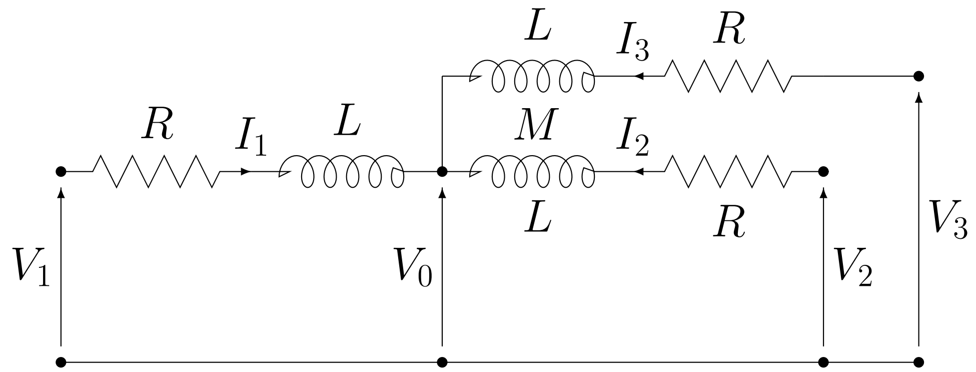
Three-phase circuit

Let us consider the following star connected three-phase circuit.



Let the three inductance currents I_1, I_2 and I_3 be the state variables of the system. Moreover, let us introduce the following matrices:

$$\mathbf{L}=\begin{bmatrix} L & M & M\\ M & L & M\\ M & M & L \end{bmatrix},\ \mathbf{R}=\begin{bmatrix} R & 0 & 0\\ 0 & R & 0\\ 0 & 0 & R \end{bmatrix},\ \mathbf{I}=\begin{bmatrix} I_1\\ I_2\\ I_3 \end{bmatrix},\ \mathbf{V}=\begin{bmatrix} V_1\\ V_2\\ V_3 \end{bmatrix},\ \mathbf{V}_0=\begin{bmatrix} V_0\\ V_0\\ V_0 \end{bmatrix}$$

The parameters L and M are the self and mutual inductance coefficients of the three inductors. The differential equation of the first inductor is obtained imposing that the time derivative of the flux $\Phi_1 = LI_1 + MI_2 + MI_3$ is equal to the input voltage $V_{L1} = V_1 - V_0 - RI_1$:

$$\frac{d\Phi_1}{dt} = V_{L1} \qquad \rightarrow \qquad L\dot{I}_1 + M\dot{I}_2 + M\dot{I}_3 = V_1 - V_0 - RI_1$$

Similar differential equations can also be written for the other two inductances. Arranging the three differential equations in a matrix form, one obtains the following matrix differential equation:

$$\begin{bmatrix} L & M & M\\ M & L & M\\ M & M & L \end{bmatrix} \begin{bmatrix} \dot{I}_1\\ \dot{I}_2\\ \dot{I}_3 \end{bmatrix} = -\begin{bmatrix} R & 0 & 0\\ 0 & R & 0\\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} I_1\\ I_2\\ I_3 \end{bmatrix} + \begin{bmatrix} V_1\\ V_2\\ V_3 \end{bmatrix} - \begin{bmatrix} V_0\\ V_0\\ V_0 \end{bmatrix}$$

which can be expressed in the following compact form:

$$\mathbf{L}\dot{\mathbf{I}} = -\mathbf{R}\mathbf{I} + \mathbf{V} - \mathbf{V}_0.$$

This is a particular case of a Power-Oriented Graph (POG) dynamic system:

$$\mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

when $\mathbf{x} = \mathbf{I}$ and $\mathbf{B}\mathbf{u} = \mathbf{V} - \mathbf{V}_0$.

Definition. *A* **"Power-Oriented Graph dynamic system"** $\mathbf{S}$ *is characterized by a state space differential equation having the following structure:*

$$\mathbf{S} = \begin{cases} \mathbf{L}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

where $\mathbf{L}$ *is the* **energy matrix**, $\mathbf{A}$ *is the* **power matrix**, $\mathbf{B}$ *is the* **input power matrix**, $\mathbf{C}$ *is the* **output matrix** *and* $\mathbf{D}$ *is the* **input-output matrix**. *Moreover, a POG dynamic system satisfies the following properties:*

a) matrix $\mathbf{L} = \mathbf{L}^T \geq 0$ *is a symmetric semidefinite matrix;*

b) the energy E_s *stored in the system can be expressed as follows:*

$$E_s = \frac{1}{2}\mathbf{x}^T\mathbf{L}\mathbf{x} \geq 0$$

c) the power P_d *dissipated in the system can be expressed as follows:*

$$P_d = \mathbf{x}^T\left(\mathbf{A} + \frac{\dot{\mathbf{L}}}{2}\right)\mathbf{x}.$$

Property. *A POG dynamic system* $\mathbf{S}$ *can be transformed in the state space using a* ***"congruent"*** *state space transformation* $\mathbf{x} = \mathbf{T}\,\mathbf{z}$*:*

$$\mathbf{S} = \begin{cases} \mathbf{L}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u} \end{cases} \quad \stackrel{\mathbf{x}=\mathbf{T}\,\mathbf{z}}{\Longrightarrow} \quad \overline{\mathbf{S}} = \begin{cases} \overline{\mathbf{L}}\,\dot{\mathbf{z}} = \overline{\mathbf{A}}\,\mathbf{z} + \overline{\mathbf{B}}\,\mathbf{u} \\ \mathbf{y} = \overline{\mathbf{C}}\,\mathbf{z} + \mathbf{D}\,\mathbf{u} \end{cases}$$

The transformed system $\overline{\mathbf{S}}$ *is characterized by the following matrices:*

$$\overline{\mathbf{L}} = \mathbf{T}^T\mathbf{L}\mathbf{T}, \qquad \overline{\mathbf{A}} = \mathbf{T}^T\mathbf{A}\mathbf{T} - \mathbf{T}^T\mathbf{L}\dot{\mathbf{T}}, \qquad \overline{\mathbf{B}} = \mathbf{T}^T\mathbf{B}, \qquad \overline{\mathbf{C}} = \mathbf{C}\mathbf{T}$$

where $\dot{\mathbf{T}}$ *is the time derivative of matrix* $\mathbf{T}$. *The transformed system* $\overline{\mathbf{S}}$ *is a "POG dynamic system" such that:*

$$\overline{\mathbf{L}} = \overline{\mathbf{L}}^T \geq 0, \qquad \bar{E}_s = \frac{1}{2}\overline{\mathbf{x}}^T\overline{\mathbf{L}}\overline{\mathbf{x}} = E_s \qquad \bar{P}_d = \overline{\mathbf{x}}^T\left(\overline{\mathbf{A}} + \frac{\dot{\overline{\mathbf{L}}}}{2}\right)\overline{\mathbf{x}} = P_d.$$

The POG systems $\mathbf{S}$ *and* $\overline{\mathbf{S}}$ *share the same energy* E_s *and the same power* P_d.

Proof. *Substituting $\mathbf{x} = \mathbf{T}\overline{\mathbf{x}}$ in the POG system $\mathbf{S}$, one obtains:*

$$\left\{ \begin{array}{r} \mathbf{L}(\mathbf{T}\dot{\overline{\mathbf{x}}} + \dot{\mathbf{T}}\overline{\mathbf{x}}) = \mathbf{A}\mathbf{T}\overline{\mathbf{x}} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{T}\overline{\mathbf{x}} + \mathbf{D}\mathbf{u} \end{array} \right. \quad \rightarrow \quad \left\{ \begin{array}{r} \mathbf{L}\mathbf{T}\dot{\overline{\mathbf{x}}} = (\mathbf{A}\mathbf{T} - \mathbf{L}\dot{\mathbf{T}})\overline{\mathbf{x}} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{T}\overline{\mathbf{x}} + \mathbf{D}\mathbf{u} \end{array} \right.$$

Left multiplying the first equation by matrix $\mathbf{T}^T$, one obtains:

$$\left\{ \begin{array}{l} \overbrace{\mathbf{T}^T\mathbf{L}\mathbf{T}}^{\overline{\mathbf{L}}}\,\dot{\overline{\mathbf{x}}} = \overbrace{(\mathbf{T}^T\mathbf{A}\mathbf{T} - \mathbf{T}^T\mathbf{L}\dot{\mathbf{T}})}^{\overline{\mathbf{A}}}\,\overline{\mathbf{x}} + \overbrace{\mathbf{T}^T\mathbf{B}}^{\overline{\mathbf{B}}}\,\mathbf{u} \\ \qquad\quad \mathbf{y} = \underbrace{\mathbf{C}\mathbf{T}}_{\overline{\mathbf{C}}}\,\overline{\mathbf{x}} + \mathbf{D}\mathbf{u} \end{array} \right.$$

The transformed system $\overline{\mathbf{S}}$ satisfy the properties of a POG dynamic system:

a) matrix $\overline{\mathbf{L}} = \overline{\mathbf{L}}^T \geq 0$ is a symmetric semidefinite matrix:

$$\overline{\mathbf{L}}^T = (\mathbf{T}^T\mathbf{L}\mathbf{T})^T = \mathbf{T}^T\mathbf{L}^T\mathbf{T} = \mathbf{T}^T\mathbf{L}\mathbf{T} = \overline{\mathbf{L}} \geq 0$$

b) the energy $\bar{E}_s$ stored in the system is :

$$\bar{E}_s = \frac{1}{2}\overline{\mathbf{x}}^T\overline{\mathbf{L}}\overline{\mathbf{x}} = \frac{1}{2}\overline{\mathbf{x}}^T\overline{\mathbf{T}}^T\mathbf{L}\mathbf{T}\overline{\mathbf{x}} = \frac{1}{2}\mathbf{x}^T\mathbf{L}\mathbf{x} = E_s \geq 0$$

c) the power $\bar{P}_d$ dissipated in the system is:

$$\begin{aligned} \bar{P}_d &= \overline{\mathbf{x}}^T\left(\overline{\mathbf{A}} + \tfrac{\dot{\overline{\mathbf{L}}}}{2}\right)\overline{\mathbf{x}} \\ &= \overline{\mathbf{x}}^T\left(\mathbf{T}^T\mathbf{A}\mathbf{T} - \mathbf{T}^T\mathbf{L}\dot{\mathbf{T}} + \tfrac{1}{2}(\dot{\mathbf{T}}^T\mathbf{L}\mathbf{T} + \mathbf{T}^T\mathbf{L}\dot{\mathbf{T}})\right)\overline{\mathbf{x}} \\ &= \overline{\mathbf{x}}^T\mathbf{T}^T\mathbf{A}\mathbf{T}\overline{\mathbf{x}} + \tfrac{1}{2}\underbrace{\overline{\mathbf{x}}^T(\dot{\mathbf{T}}^T\mathbf{L}\mathbf{T} - \mathbf{T}^T\mathbf{L}\dot{\mathbf{T}})\overline{\mathbf{x}}}_{0} \\ &= \mathbf{x}^T\mathbf{A}\mathbf{x} = P_d \end{aligned}$$

Example. For the given system:

$$\mathbf{L}\dot{\mathbf{I}} = \underbrace{-\mathbf{R}}_{\mathbf{A}}\,I + \underbrace{\mathbf{V} - \mathbf{V}_0}_{\mathbf{B}\mathbf{u}} \qquad \leftrightarrow \qquad \mathbf{L}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

it is:

$$\begin{aligned} E_s &= \frac{1}{2}\mathbf{x}^T\mathbf{L}\mathbf{x} = \frac{L}{2}(I_1^2 + I_2^2 + I_3^2) + M(I_1I_2 + I_1I_3 + I_2I_2) \\ P_d &= \mathbf{x}^T\left(\mathbf{A} + \frac{\dot{\mathbf{L}}}{2}\right)\mathbf{x} = -R(I_1^2 + I_2^2 + I_3^2) \end{aligned}$$

Property. *For the POG dynamic systems the transfer matrices* $\mathbf{H}(s)$ *and* $\mathbf{H}(z)$ *of continuous-time and discrete-time systems:*

$$\begin{cases} \mathbf{L}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t) \end{cases} \qquad \begin{cases} \mathbf{L}\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k)+\mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)+\mathbf{D}\mathbf{u}(k) \end{cases}$$

can be obtained as follows:

$$\mathbf{H}(s) = \mathbf{C}(\mathbf{L}\,s-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}, \qquad \mathbf{H}(z) = \mathbf{C}(\mathbf{L}\,z-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}.$$

Proof. For the continuous-time systems the POG dynamic system equations can be rewritten as follows:

$$\begin{cases} \dot{\mathbf{x}}(t) = \underbrace{\mathbf{L}^{-1}\mathbf{A}}_{\overline{\mathbf{A}}}\mathbf{x}(t)+\underbrace{\mathbf{L}^{-1}\mathbf{B}}_{\overline{\mathbf{B}}}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t) \end{cases}$$

and the transfer matrix $\mathbf{H}(s)$ is:

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{C}(\mathbf{I}\,s-\overline{\mathbf{A}})^{-1}\overline{\mathbf{B}}+\mathbf{D} \\ &= \mathbf{C}(\mathbf{I}\,s-(\mathbf{L}^{-1}\mathbf{A}))^{-1}(\mathbf{L}^{-1}\mathbf{B})+\mathbf{D} \\ &= \mathbf{C}(\mathbf{L}^{-1}(\mathbf{L}\,s-\mathbf{A}))^{-1}(\mathbf{L}^{-1}\mathbf{B})+\mathbf{D} \\ &= \mathbf{C}(\mathbf{L}\,s-\mathbf{A})^{-1}\,\mathbf{L}\,(\mathbf{L}^{-1}\mathbf{B})+\mathbf{D} \\ &= \mathbf{C}(\mathbf{L}\,s-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} \end{aligned}$$

Property. *A time-invariant "congruent" state space transformation* $\mathbf{x} = \mathbf{T}\,\mathbf{z}$ *applied to a POG dynamic system* $\mathbf{S}$

$$\mathbf{S} = \begin{cases} \mathbf{L}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}+\mathbf{B}\,\mathbf{u} \\ \mathbf{y} = \mathbf{C}\,\mathbf{x}+\mathbf{D}\,\mathbf{u} \end{cases} \quad \overset{\mathbf{x}=\mathbf{T}\,\mathbf{z}}{\Longrightarrow} \quad \overline{\mathbf{S}} = \begin{cases} \overline{\mathbf{L}}\,\dot{\mathbf{z}} = \overline{\mathbf{A}}\,\mathbf{z}+\overline{\mathbf{B}}\,\mathbf{u} \\ \mathbf{y} = \overline{\mathbf{C}}\,\mathbf{z}+\mathbf{D}\,\mathbf{u} \end{cases}$$

does not modify the transfer matrices $\mathbf{H}(s)$ *of the system:*

$$\mathbf{H}(s) = \overline{\mathbf{C}}(\overline{\mathbf{L}}\,s-\overline{\mathbf{A}})^{-1}\overline{\mathbf{B}}+\mathbf{D} = \mathbf{C}(\mathbf{L}\,s-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

Proof. The transfer matrix $\mathbf{H}(s)$ is:

$$\begin{aligned} \mathbf{H}(s) &= \overline{\mathbf{C}}(\overline{\mathbf{L}}\,s-\overline{\mathbf{A}})^{-1}\overline{\mathbf{B}}+\mathbf{D} \\ &= \mathbf{C}\mathbf{T}(\mathbf{T}^T\mathbf{L}\mathbf{T}s-\mathbf{T}^T\mathbf{A}\mathbf{T})^{-1}\mathbf{T}^T\mathbf{B}+\mathbf{D} \\ &= \mathbf{C}\mathbf{T}(\mathbf{T}^T(\mathbf{L}s-\mathbf{A})\mathbf{T})^{-1}\mathbf{T}^T\mathbf{B}+\mathbf{D} \\ &= \mathbf{C}\mathbf{T}\mathbf{T}^{-1}(\mathbf{L}s-\mathbf{A})^{-1}\mathbf{T}^{-T}\mathbf{T}^T\mathbf{B}+\mathbf{D} \\ &= \mathbf{C}(\mathbf{L}\,s-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} \end{aligned}$$

The state variables I_1, I_2 and I_3 are constrained as follows:

$$I_1 + I_2 + I_3 = 0.$$

This constraint can be expressed in a vectorial form:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = 0 \qquad \leftrightarrow \qquad \mathbf{v}_3^T \mathbf{I} = 0 \qquad \text{where} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The state vector $\mathbf{I}(t)$ changes in time, but always remains perpendicular to vector $\mathbf{v}_3$. The best way of "inserting" the constraint $\mathbf{v}_3^T\mathbf{I} = 0$ within the mathematical equations of the system is to use the state space transformation $\mathbf{I} = \overline{\mathbf{T}}\,\overline{\mathbf{I}}$ where $\overline{\mathbf{v}}_3$ is one of the vectors of the new system basis:

$$\overline{\mathbf{T}} = \begin{bmatrix} \overline{\mathbf{v}}_1 & \overline{\mathbf{v}}_2 & \overline{\mathbf{v}}_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \qquad \overline{\mathbf{v}}_3 = \frac{\mathbf{v}_3}{|\mathbf{v}_3|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

The two vectors $\overline{\mathbf{v}}_1$ and $\overline{\mathbf{v}}_2$ are chosen in such a way that the transformation $\overline{\mathbf{T}}$ is orthonormal. With this choice the inverse $\overline{\mathbf{T}}^{-1}$ of matrix $\mathbf{T}$ is equal to the transpose of matrix $\overline{\mathbf{T}}$:

$$\overline{\mathbf{T}}^{-1} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \overline{\mathbf{T}}^T$$

Applying to system:

$$\mathbf{L}\dot{\mathbf{I}} = -\mathbf{R}\mathbf{I} + \mathbf{V} - \mathbf{V}_0$$

the "congruent" state space transformation $\mathbf{I} = \overline{\mathbf{T}}\,\overline{\mathbf{I}}$ one obtains:

$$\underbrace{(\overline{\mathbf{T}}^T\mathbf{L}\overline{\mathbf{T}})}_{\overline{\mathbf{L}}}\,\dot{\overline{\mathbf{I}}} = -\underbrace{(\overline{\mathbf{T}}^T\mathbf{R}\overline{\mathbf{T}})}_{\overline{\mathbf{R}}}\,\overline{\mathbf{I}} + \overline{\mathbf{T}}^T(\mathbf{V} - \mathbf{V}_0)$$

The transformed matrix $\overline{\mathbf{R}}$ is diagonal and equal to $\mathbf{R}$:

$$\overline{\mathbf{R}} = \overline{\mathbf{T}}^T\mathbf{R}\overline{\mathbf{T}} = \mathbf{R}.$$

The transformed matrix $\overline{\mathbf{L}} = \overline{\mathbf{T}}^T\mathbf{L}\overline{\mathbf{T}}$ has the following diagonal structure:

$$\overline{\mathbf{L}} = \overline{\mathbf{T}}^T\mathbf{L}\overline{\mathbf{T}} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} L & M & M \\ M & L & M \\ M & M & L \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{2(L-M)}{\sqrt{6}} & 0 & \frac{(L+2M)}{\sqrt{3}} \\ \frac{-(L-M)}{\sqrt{6}} & \frac{(L-M)}{\sqrt{2}} & \frac{(L+2M)}{\sqrt{3}} \\ \frac{-(L-M)}{\sqrt{6}} & \frac{-(L-M)}{\sqrt{2}} & \frac{(L+2M)}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} (L-M) & 0 & 0 \\ 0 & (L-M) & 0 \\ 0 & 0 & (L+2M) \end{bmatrix}$$

Computations in Matlab:

```
-- Matlab commands ---------------------------------------------------------
syms L M R
LM = [L M M;
      M L M;
      M M L];
T = [ 2/sqrt(6)         0   1/sqrt(3);
     -1/sqrt(6)  1/sqrt(2)  1/sqrt(3);
     -1/sqrt(6) -1/sqrt(2)  1/sqrt(3)];
LT = simplify(T.'*LM*T)

-- Matlab output -----------------------------------------------------------
 LT =

 [ L - M,     0,        0]
 [     0, L - M,        0]
 [     0,     0, L + 2*M]
---------------------------------------------------------------------------
```

The transformed vectors $\overline{\mathbf{I}} = \overline{\mathbf{T}}^{-1}\mathbf{I}$, $\overline{\mathbf{V}} = \overline{\mathbf{T}}^{-1}\mathbf{V}$ and $\overline{\mathbf{V}}_0 = \overline{\mathbf{T}}^{-1}\mathbf{V}_0$ have the following structure:

$$\overline{\mathbf{I}} = \begin{bmatrix} \frac{2I_1-I_2-I_3}{\sqrt{6}} \\ \frac{I_2-I_3}{\sqrt{2}} \\ \frac{I_1+I_2+I_3}{\sqrt{3}} \end{bmatrix}, \qquad \overline{\mathbf{V}} = \begin{bmatrix} \frac{2V_1-V_2-V_3}{\sqrt{6}} \\ \frac{V_2-V_3}{\sqrt{2}} \\ \frac{V_1+V_2+V_3}{\sqrt{3}} \end{bmatrix}, \qquad \overline{\mathbf{V}}_0 = \begin{bmatrix} 0 \\ 0 \\ \frac{3V_0}{\sqrt{3}} \end{bmatrix}$$

```
-- Matlab commands -------------------------------------------------------------------
syms V1 V2 V3
V = [V1; V2; V3];                                          VT =
T = [ 2/sqrt(6)         0   1/sqrt(3);
     -1/sqrt(6)  1/sqrt(2)  1/sqrt(3);       -->      -(6^(1/2)*(V2 - 2*V1 + V3))/6
     -1/sqrt(6) -1/sqrt(2)  1/sqrt(3)];                        (2^(1/2)*(V2 - V3))/2
VT = simplify(T.'*V)                                      (3^(1/2)*(V1 + V2 + V3))/3
-------------------------------------------------------------------------------------
```

Being $I_1+I_2+I_3=0$, the third element $\bar{I}_3$ of the new state vector $\bar{\mathbf{I}}$ is always equal to zero: $\bar{I}_3=0$:

$$\bar{\mathbf{I}} = \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} \frac{2I_1-I_2-I_3}{\sqrt{6}} \\ \frac{I_2-I_3}{\sqrt{2}} \\ 0 \end{bmatrix}$$

The new state space equations of the given system are now the following:

$$\begin{bmatrix} L-M & 0 & 0 \\ 0 & L-M & 0 \\ 0 & 0 & L+2M \end{bmatrix} \begin{bmatrix} \dot{\bar{I}}_1 \\ \dot{\bar{I}}_2 \\ 0 \end{bmatrix} = -\begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{2V_1-V_2-V_3}{\sqrt{6}} \\ \frac{V_2-V_3}{\sqrt{2}} \\ \frac{V_1+V_2+V_3-3V_0}{\sqrt{3}} \end{bmatrix}$$

The third system equation is a static constraint which can be used to determine the value V_0 of the voltage of the center of the star connected circuit:

$$V_0 = \frac{V_1+V_2+V_3}{3}.$$

So, the considered three-phases system has a "second order" internal dynamics:

$$\overbrace{\begin{bmatrix} L-M & 0 \\ 0 & L-M \end{bmatrix}}^{\mathbf{L}_b} \overbrace{\begin{bmatrix} \dot{\bar{I}}_1 \\ \dot{\bar{I}}_2 \end{bmatrix}}^{\dot{\mathbf{I}}_b} = -\overbrace{\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}}^{\mathbf{R}_b} \overbrace{\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix}}^{\mathbf{I}_b} + \overbrace{\begin{bmatrix} \frac{2V_1-V_2-V_3}{\sqrt{6}} \\ \frac{V_2-V_3}{\sqrt{2}} \end{bmatrix}}^{\mathbf{V}_b}$$

In compact form the system dynamics can be written as follows:

$$\mathbf{L}_b\,\dot{\mathbf{I}}_b = -\mathbf{R}\,\mathbf{I}_b + \mathbf{V}_b \qquad \leftrightarrow \qquad (L-M)\,\dot{\mathbf{I}}_b = -R\,\mathbf{I}_b + \mathbf{V}_b$$

Let us now suppose that the input voltages are "balanced":

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = V_M \begin{bmatrix} \cos(\theta) \\ \cos(\theta-\frac{2\pi}{3}) \\ \cos(\theta+\frac{2\pi}{3}) \end{bmatrix}$$

and that the amplitude V_M of the voltages V_1, V_2 and V_3 is the only degree of freedom for controlling the system. With this choice, the three voltages V_1, V_2 and V_3 satisfy the same algebraic constraint used for the currents:

$$V_1+V_2+V_3=0 \qquad \leftrightarrow \qquad \mathbf{v}_3^T\mathbf{V}=0$$

The third element of vector $\overline{\mathbf{V}}$ is always equal to zero. From which it follows that $V_0 = 0$. The vector $\mathbf{V}_b$ as the following form:

$$\mathbf{V}_b = \begin{bmatrix} \frac{2V_1-V_2-V_3}{\sqrt{6}} \\ \frac{V_2-V_3}{\sqrt{2}} \end{bmatrix} = V_M \begin{bmatrix} \frac{2\cos(\theta)-\cos(\theta-\frac{2\pi}{3})-\cos(\theta+\frac{2\pi}{3})}{\sqrt{6}} \\ \frac{\cos(\theta-\frac{2\pi}{3})-\cos(\theta+\frac{2\pi}{3})}{\sqrt{2}} \end{bmatrix}$$

$$= V_M \begin{bmatrix} \frac{3\cos(\theta)}{\sqrt{6}} \\ \frac{-2\sin(\theta)\sin(\frac{-2\pi}{3})}{\sqrt{2}} \end{bmatrix} = V_M \sqrt{\frac{3}{2}} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

```
-- Matlab commands -----------------------------------------------------------------
 syms V1 V2 V3
 V = [V1; V2; V3];                        Vb =
 V1=cos(th);
 V2=cos(th-2*pi/3);        -->              (6^(1/2)*cos(th))/2
 V3=cos(th+2*pi/3);                         (6^(1/2)*sin(th))/2
 Vb = simplify(eval(VT))                                      0
-----------------------------------------------------------------------------------
```

The obtained system is "linear" and *"time-varying"* because the input matrix $\mathbf{B}_b$ is a function of parameter θ:

$$\underbrace{\begin{bmatrix} L-M & 0 \\ 0 & L-M \end{bmatrix}}_{\mathbf{L}_b} \underbrace{\begin{bmatrix} \dot{\bar{I}}_1 \\ \dot{\bar{I}}_2 \end{bmatrix}}_{\dot{\mathbf{I}}_b} = -\underbrace{\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}}_{\mathbf{R}_b} \underbrace{\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix}}_{\mathbf{I}_b} + \underbrace{\sqrt{\frac{3}{2}} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}}_{\mathbf{B}_b} V_M$$

In compact form one obtains the following differential equation:

$$\mathbf{L}_b \, \dot{\mathbf{I}}_b = -\mathbf{R}_b \, \mathbf{I}_b + \mathbf{B}_b \, V_M \qquad (1)$$

The eigenvalues of this system are real: $\lambda_{1,2} = \frac{-R}{L-M}$.

If $\theta = \omega t$ it is possible to apply to system (1) the following new "congruent" state space transformation $\mathbf{I}_b = \mathbf{T}_\omega \mathbf{I}_\omega$ where:

$$\mathbf{T}_\omega = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}, \qquad \mathbf{T}_\omega^{-1} = \mathbf{T}_\omega^T = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}.$$

The two matrix $\mathbf{T}_\omega$ and $\mathbf{T}_\omega^{-1}$ represent two rotations in the plane of angles $\theta = \omega t$ and $\theta = -\omega t$, respectively. Applying the congruent transformation $\mathbf{I}_b = \mathbf{T}_\omega \mathbf{I}_\omega$ to system (1) one obtains the following POG transformed system:

$$\underbrace{\mathbf{T}_\omega^T \mathbf{L}_b \mathbf{T}_\omega}_{\mathbf{L}_\omega} \dot{\mathbf{I}}_\omega = (-\underbrace{\mathbf{T}_\omega^T \mathbf{R}_b \mathbf{T}_\omega}_{\mathbf{R}_\omega} - \underbrace{\mathbf{T}_\omega^T \mathbf{L}_b \dot{\mathbf{T}}_\omega}_{\mathbf{A}_\omega}) \, \mathbf{I}_\omega + \underbrace{\mathbf{T}_\omega^T \mathbf{B}_b}_{\mathbf{B}_\omega} V_M$$

Matrix $\mathbf{T}_\omega(\theta)$ is time-varying and therefore in the transformed system it is present also the additional term $\mathbf{A}_\omega = \mathbf{T}_\omega^T \mathbf{L}_b \dot{\mathbf{T}}_\omega$. The transformed matrix $\mathbf{A}_\omega$ has the following form:

$$\mathbf{A}_\omega = \mathbf{T}_\omega^{-1} \mathbf{L}_b \dot{\mathbf{T}}_\omega = \begin{bmatrix} 0 & -\omega(L-M) \\ \omega(L-M) & 0 \end{bmatrix}$$

The transformed vector $\mathbf{B}_\omega$ has the following form:

$$\mathbf{B}_\omega = \mathbf{T}_\omega^{-1} \mathbf{B}_b = \mathbf{T}_\omega^{-1} \sqrt{\frac{3}{2}} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{3}{2}} \\ 0 \end{bmatrix}$$

The transformed vector $\mathbf{V}_\omega = \mathbf{B}_\omega V_M$ is now constant:

$$\mathbf{V}_\omega = \mathbf{B}_\omega V_M = V_M \begin{bmatrix} \sqrt{\frac{3}{2}} \\ 0 \end{bmatrix}$$

The final POG transformed system is "linear" and **"time-invariant"**:

$$\left\{ \begin{array}{rcl} \dot{\bar{\mathbf{I}}}_\omega & = & \begin{bmatrix} \frac{-R}{L-M} & \omega \\ -\omega & \frac{-R}{L-M} \end{bmatrix} \bar{\mathbf{I}}_\omega + \begin{bmatrix} \sqrt{\frac{3}{2}} \frac{1}{(L-M)} \\ 0 \end{bmatrix} V_M \\ \mathbf{I} & = & \mathbf{T}_b \mathbf{T}_\omega \bar{\mathbf{I}}_\omega \end{array} \right. \qquad (2)$$

where $\mathbf{T}_b$ denotes the rectangular matrix obtained using the first two columns of matrix $\overline{\mathbf{T}}$:

$$\mathbf{T}_b = \begin{bmatrix} \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

The eigenvalues of system (2) are complex conjugate:

$$\lambda_{1,2} = \frac{-R}{L-M} \pm j\omega$$

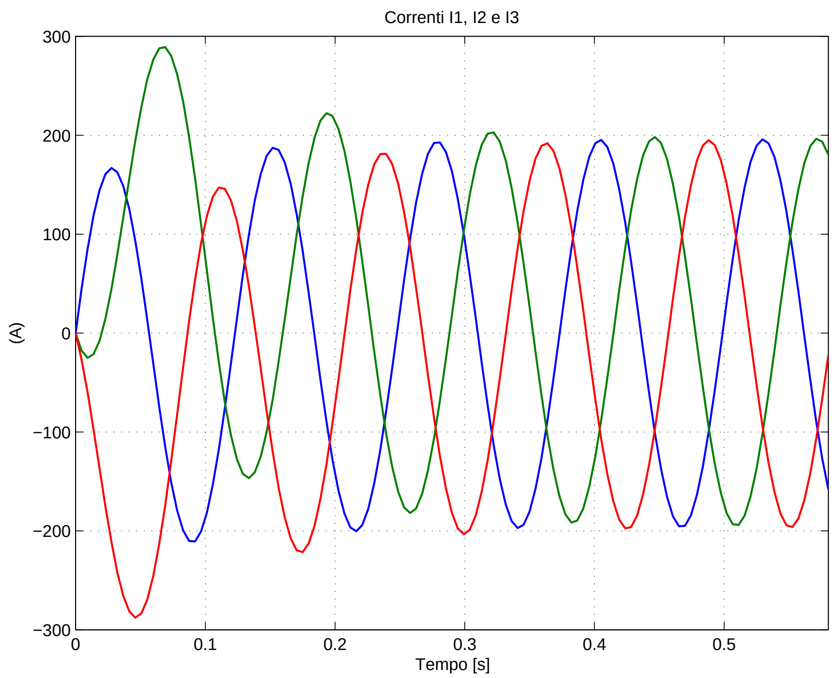
The time-varying state space transformation $\mathbf{I}_b = \mathbf{T}_\omega \mathbf{I}_\omega$ has changed "the imaginary part of the system poles". The equations of the POG dynamic system (2) are completely equivalent to the initial differential equations:

$$\begin{bmatrix} L & M & M \\ M & L & M \\ M & M & L \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \end{bmatrix} = - \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} - \begin{bmatrix} V_0 \\ V_0 \\ V_0 \end{bmatrix}$$

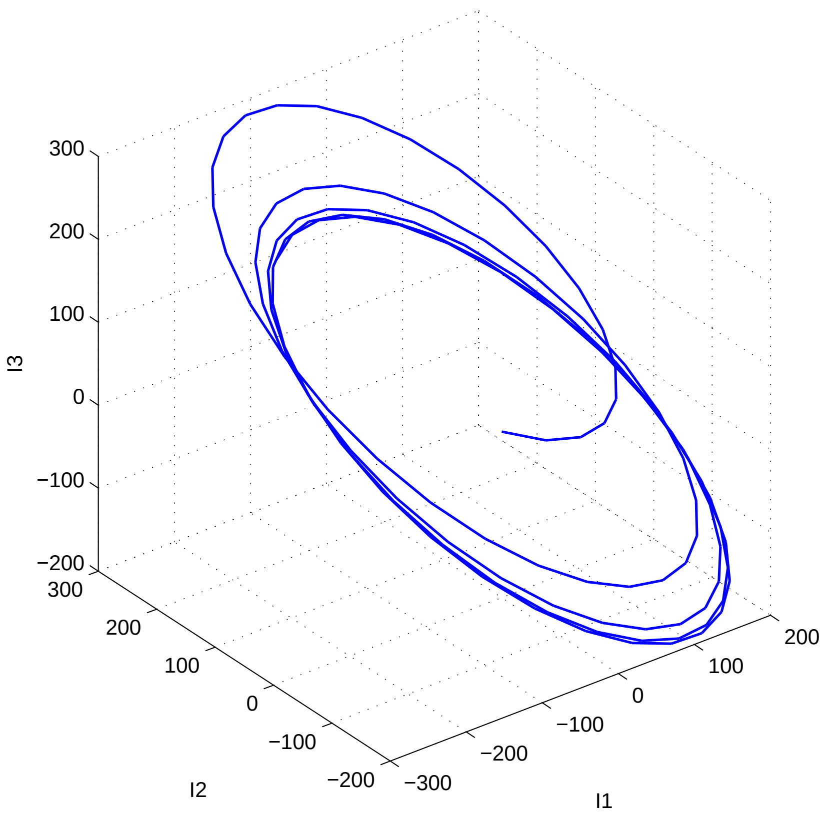
with the constraint $I_1 + I_2 + I_3 = 0$.

Simulations: static three-phase state space

- Time behavior of the currents $\mathbf{I}$. Step response $V_M = 100$:

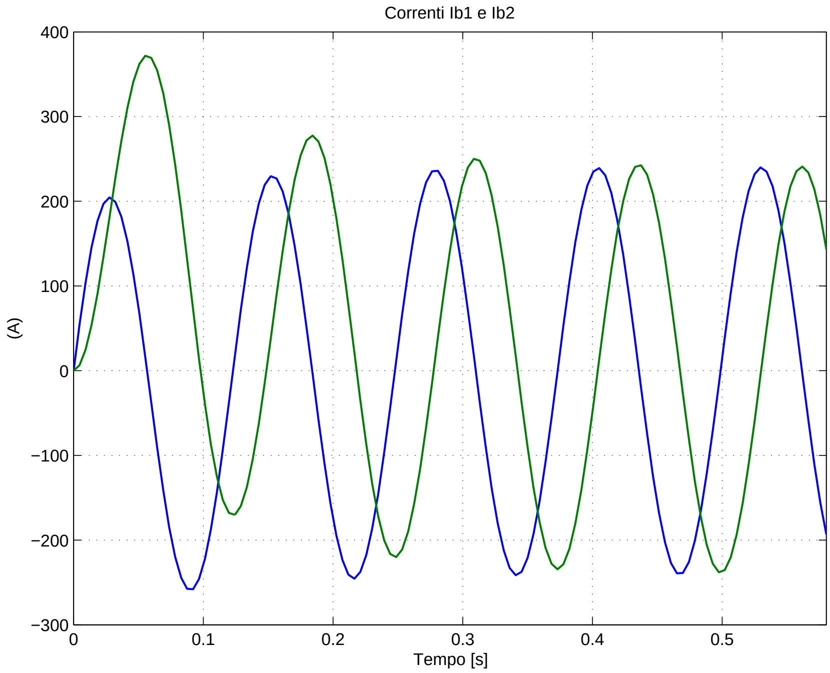


- State space trajectory:

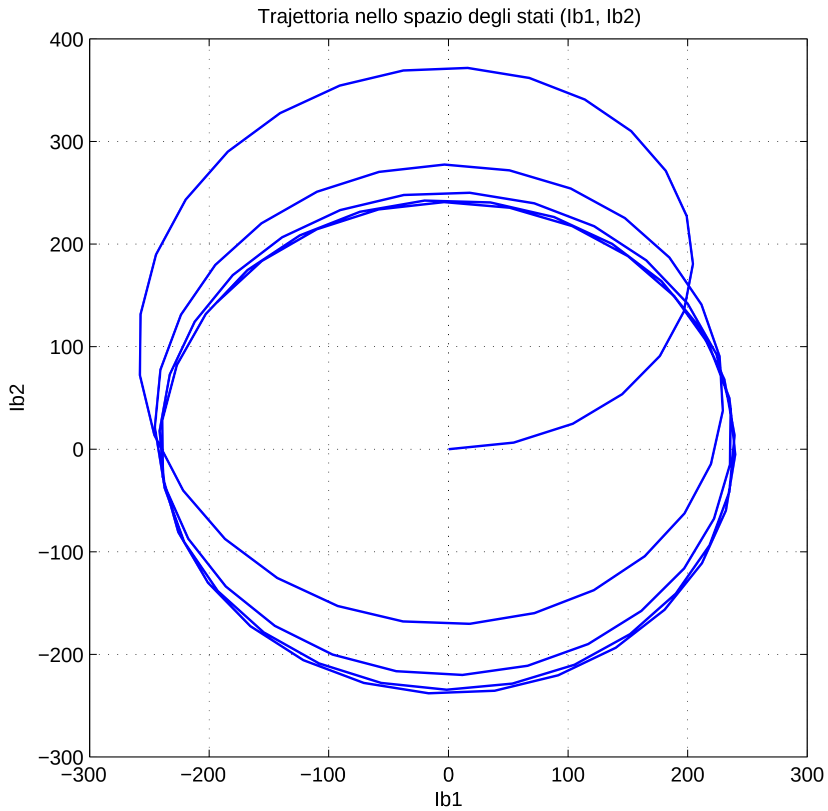


Static two-phase state space

- Time behavior of the currents $\mathbf{I}_b$. Step response $V_M = 100$:

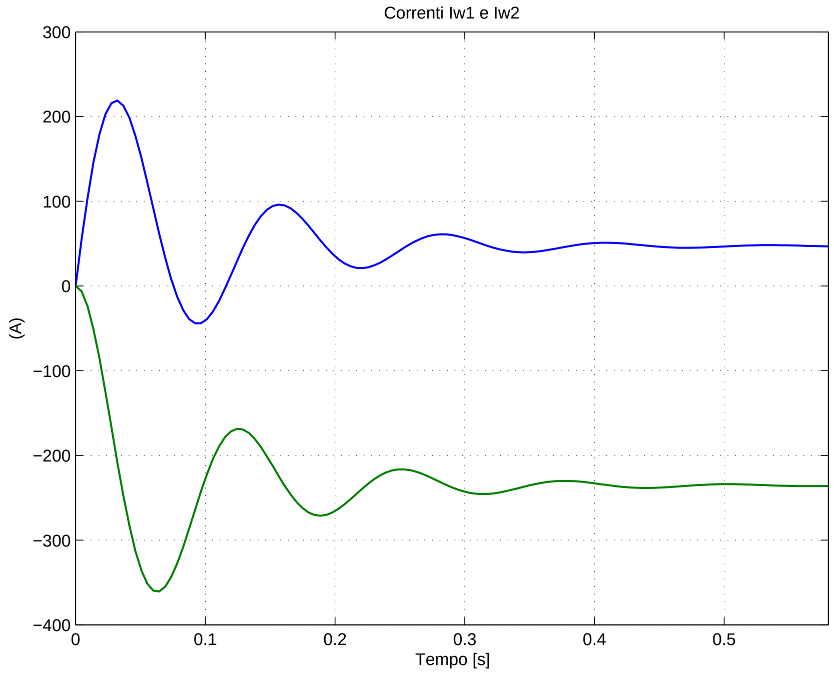


- State space trajectory:

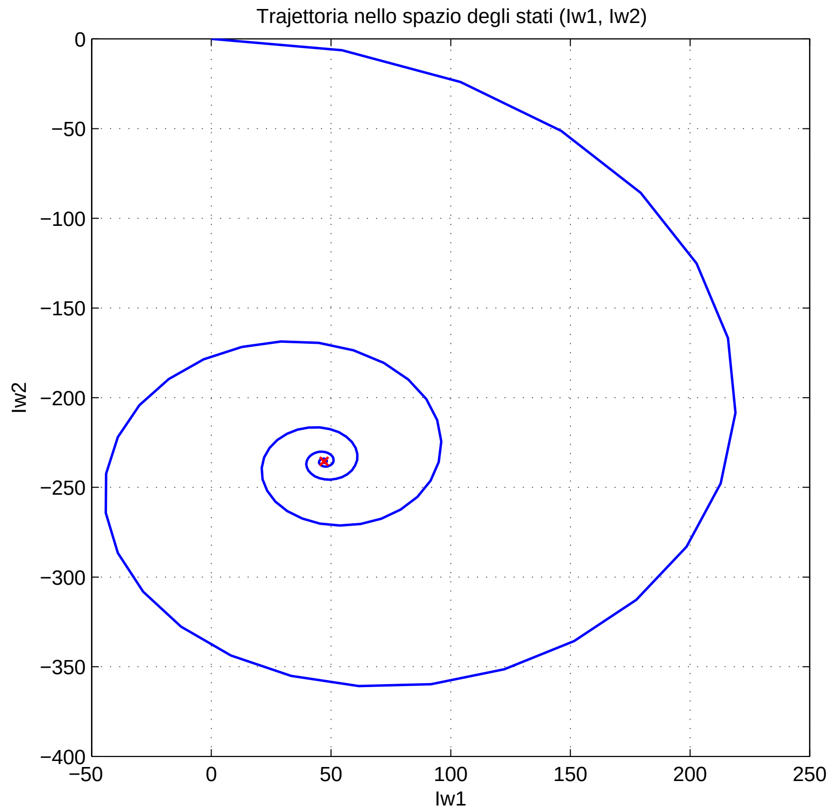


Rotating two-phase state space

- Time behavior of the currents $\mathbf{I}_\omega$. Step response $V_M = 100$:



- State space trajectory:



• Matlab commands:

```
------------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Circuito_trifase.m
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all; clc; echo off
set(0,'DefaultFigureWindowStyle','docked') %%%% Figures 'docked' or 'normal'
MainString = mfilename;     % MainString
%%%%% Unit di misura
A=1; ohm=1; V=1; H=1; rad=1; sec=1; Stampa=1;
%%%%% Parametri del sistema
R=0.1*ohm;                  % Phase resistance
L=0.05*H;                   % Phase self-inductance coefficient
M=0.04*H;                   % Phase mutual-inductance coefficient
w=50*rad/sec;               % Electric frequency
VM=100*V;                   % Amplitude of the input voltages
%%%%% System matrices
MA=[ -R/(L-M)       w;
          -w  -R/(L-M)];
MB=[  sqrt(3/2)/(L-M); 0];
MC=[  1   0; 0   1];
SYS=ss(MA,MB,MC,0);         % System SYS in the rotating frame
set(SYS,'InputName','V_M')
set(SYS,'StateName',['Iw_1'; 'Iw_2'])
set(SYS,'OutputName',['Iw_1'; 'Iw_2'])
SYS
%%%%% Ste respose of system SYS
[Y t X]=step(SYS);
X=X*VM;                 % The system response is linear with respect ...
Y=Y*VM;                 % ... to the amplitude of the input voltage VM
Iw1=X(:,1);             % Rotating two-phase current: first term
Iw2=X(:,2);             % Rotating two-phase current: second term
Lw=1.2;                 % Line width
%
%%%%% Time behaviors of currents Iw1 Iw2
figure(1); clf
plot(t,[Iw1 Iw2],'LineWidth',Lw)
xlim([0 t(end)])
title('Currents Iw1 and Iw2')
xlabel('Time [s]')
ylabel('(A)'); grid
if Stampa;  eval(['print -depsc ' MainString '_'  num2str(gcf)]); end
%
%%%%% Trajectory (Iw1 Iw2) in the rotating frame
figure(2); clf; hold off
plot(Iw1,Iw2,'LineWidth',Lw); hold on
Iw_ss=-inv(MA)*MB*VM;       % Steady state value of currents Iw1 and Iw2
plot(Iw_ss(1),Iw_ss(2),'rx','LineWidth',Lw)
axis square; grid
title('State space trajectories (Iw1, Iw2)')
ylabel('Iw2')
xlabel('Iw1')
if Stampa;  eval(['print -depsc ' MainString '_'  num2str(gcf)]); end
%
%%%%%  The transformation matrix Tw has the following structure:
%      Tw=[cos(w*t)  -sin(w*t);
%          sin(w*t)   cos(w*t)];
% the behavior of the currents in the two-phase static plane is the following:
Ib1=sum(([cos(w*t)  -sin(w*t)].*[Iw1';Iw2']')')';
Ib2=sum(([sin(w*t)   cos(w*t)].*[Iw1';Iw2']')')';
%
```

```
%%%%% Time behaviors of currents Ib1 Ib2
figure(3); clf
plot(t,[Ib1 Ib2],'LineWidth',Lw)
xlim([0 t(end)])
title('Currents Ib1 and Ib2')
ylabel('(A)'); grid
xlabel('Time [s]')
if Stampa;  eval(['print -depsc ' MainString '_'  num2str(gcf)]); end
%
%%%%% Trajectory (Ib1 Ib2) in the static two-phase plane
figure(4); clf; hold off
plot(Ib1,Ib2,'LineWidth',Lw)
grid; axis square
title('State space trajectory (Ib1, Ib2)')
ylabel('Ib2')
xlabel('Ib1')
if Stampa;  eval(['print -depsc ' MainString '_'  num2str(gcf)]); end
%
%%%%% The transformaton matrix Tb has the following form:
Tb=[
   2/sqrt(6)          0;
  -1/sqrt(6)  1/sqrt(2);
  -1/sqrt(6) -1/sqrt(2)];
I=Tb*[Ib1';Ib2'];
I1=I(1,:)'; I2=I(2,:)'; I3=I(3,:)';
%
%%%%% Time behavior of currents I1, I2 and I3
figure(5); clf
plot(t,I,'LineWidth',Lw)
xlim([0 t(end)])
title('Currents I1, I2 and I3')
ylabel('(A)'); grid
xlabel('Time [s]')
if Stampa;  eval(['print -depsc ' MainString '_'  num2str(gcf)]); end
%
%%%%% The trajectory (I1 I2 I3) in the static three-phase plane
figure(6); clf; hold off
plot3(I1,I2,I2,'LineWidth',Lw)
axis square; grid
zlabel('I3')
ylabel('I2')
xlabel('I1')
if Stampa;  eval(['print -depsc ' MainString '_'  num2str(gcf)]); end
return
--------------------------------------------------------------------------
```