

Definition of Dynamic System

There are various types of dynamic systems: continuous-time, discrete-time, linear, non-linear, lumped systems, distributed systems, finite states, etc. A dynamic system is always characterized by an input space U , a state space X and an output space Y on which are defined the following two functions:

- State transition function:

$$x(t) = \psi(t, t_0, x(t_0), u(\cdot))$$

where $t_0 \in \mathcal{T}$ is the initial time instant, $t \in \mathcal{T}$ is the current time instant, $x(t_0) \in X$ is the initial state, $x(t) \in X$ is the current state, $u(\cdot) \in \mathcal{U}$ is the input function defined in the range $[t_0, t]$.

- Output function:

$$y(t) = \eta(t, x(t), u(t))$$

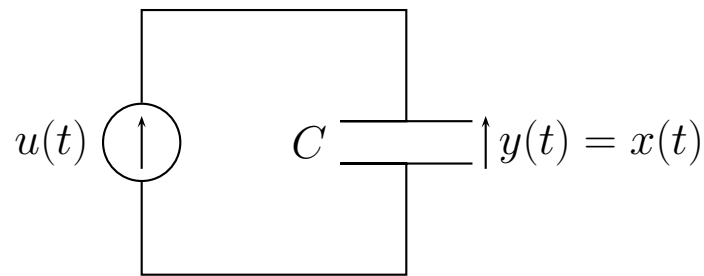
where $t \in \mathcal{T}$ is the current time instant, $x(t) \in X$ is the current state and $u(t) \in U$ is the value of the input function at the current time instant.

If the output function does not depend on the input $u(\cdot)$:

$$y(t) = \eta(t, x(t))$$

the system is “*strictly causal*” (or “*strictly proper*”). If $\mathcal{T} = \mathcal{R}$ the dynamic system is “*continuous-time*”. If $\mathcal{T} = \mathcal{Z}$ the dynamic system is “*discrete-time*”.

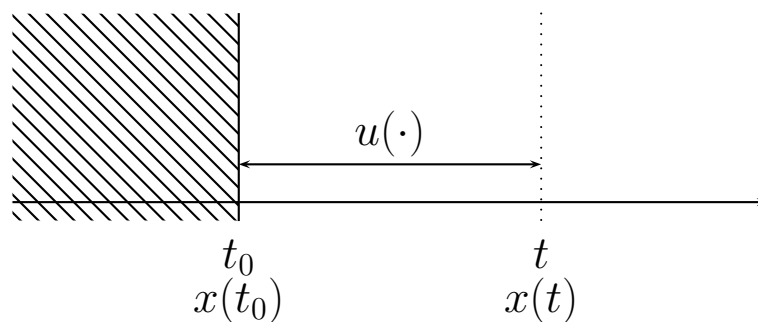
Example. Consider the dynamic system composed by a current generator $u(t)$ and a capacitor C .



The capacitor voltage $x(t)$ is the state variable of the system:

$$x(t) = x(t_0) + \frac{1}{C} \int_{t_0}^t u(\tau) d\tau = \psi(t, t_0, x(t_0), u(\cdot))$$

Physical meaning of the state vector. In a generic time instant t_0 , the state variables $x(t)$ contain all the information necessary to assess the future performance of the system, i.e. to compute the future values of the state variable $x(t)$, once the input signal $u(\cdot)$ is known.



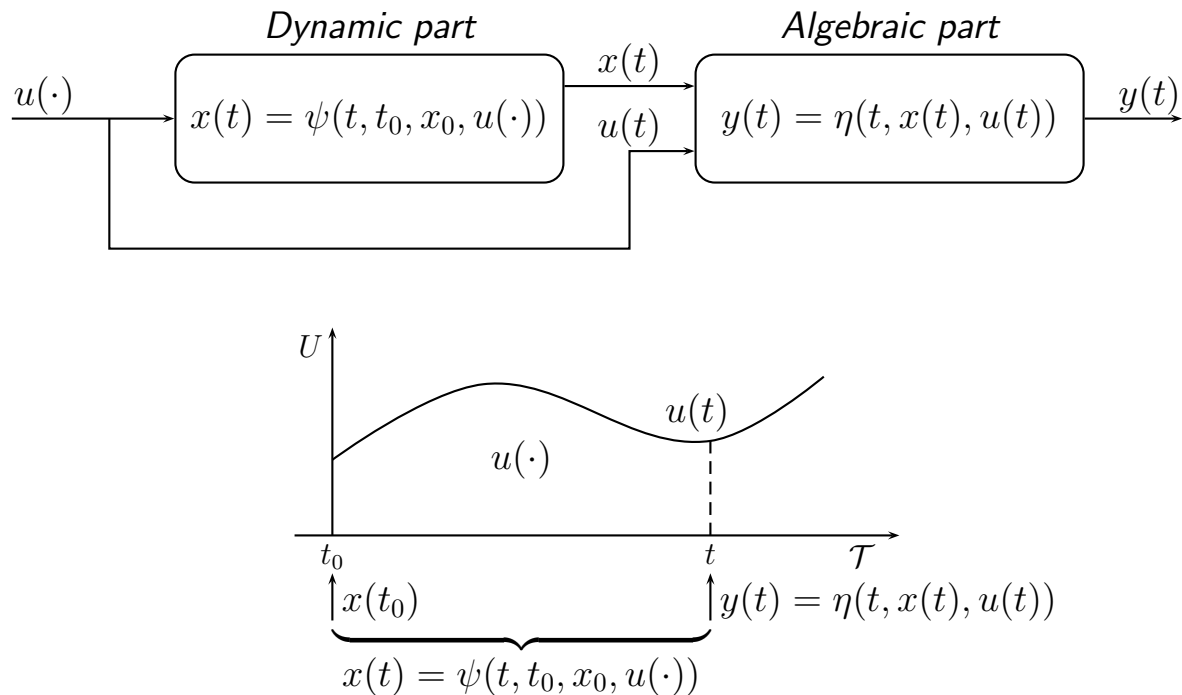
To compute state $x(t)$ at time instant t is necessary to know the state $x(t_0)$ at time t_0 and the input function $u(\cdot)$ in the range $[t_0, t]$:

$$x(t) = \psi(t, t_0, x_0, u(\cdot)).$$

The state variables allow to compute the future behavior of the system without taking into account the history of the system before the time instant t_0 .

The state variables are not “uniquely” defined: typically there are infinite many different ways to define the “state” of a system.

Separation property in the state space:



- *Dynamic part of the system.* The state transition function $\psi(\cdot)$ describes the evolution of the state variables $x(t)$ as a function of the four parameters t, t_0, x_0 and $u(\cdot)$.
- *Algebraic part of the system.* The output function $\eta(\cdot)$ describes the *output variables* as a static function of the parameters $t, x(t)$ and $u(t)$.

If the dynamic system is *continuous-time, regular* and *finite-dimensional*, the state transition function $x(t) = \psi(t, t_0, x(t_0), u(\cdot))$ is the solution of a vectorial “differential” equation:

$$\dot{x}(t) \triangleq \frac{dx(t)}{dt} = f(x(t), u(t), t)$$

In the *discrete-time* case, the state transition function $\psi(\cdot)$ is the solution of a vectorial “difference” equation:

$$x(k + 1) = f(x(k), u(k), k)$$

The functions $f(x(t), u(t), t)$ and $f(x(k), u(k), k)$, called “state functions”, implicitly describe the dynamic part of the system.

Dynamic systems: vectorial representation

- The *continuous* and *discrete-time* dynamic systems characterized by m inputs, n states and p outputs can be mathematically represented by using the following vectors:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \quad \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

where $\mathbf{x}(t) \in X = \mathcal{R}^n$ is the state vector, $\mathbf{u}(t) \in U = \mathcal{R}^m$ is the input vector, $\mathbf{y}(t) \in Y = \mathcal{R}^p$ output vector and $t \in \mathcal{T} = \mathcal{R}$ is the time variable.

- The *continuous-time time-variant* dynamic systems are “implicitly” described by a system of n first order nonlinear “*differential*” equations:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \end{cases}$$

- For *discrete-time* systems the following system of “*difference*” is used:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k), k) \\ \mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k), k) \end{cases}$$

- For *continuous [discrete]-time* and **time-invariant** systems, the state transition function $\psi(t, t_0, x(t_0), u(\cdot))$ does not depend on both the parameters t and t_0 [k and h], but it depends only on the difference $t - t_0$ [$k - h$] between the final time instant t [k] and the initial time instant t_0 [h].

For this reason, for *time-invariant* systems it is always possible to simplify the mathematical notations by substituting the initial time instant t_0 [h] with the origin of the time variable $t_0 = 0$ [$h = 0$].

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases} \quad \begin{cases} \mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) \\ \mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k)) \end{cases}$$

Linear, continuous-time dynamic systems

A non-linear, continuous-time, finite-dimensional and time-variant dynamic system can always be described in the following “implicit” form:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \end{cases}$$

If the dynamic system is linear and time-variant, the *state function* $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ and the *output function* $\mathbf{g}(\mathbf{x}, \mathbf{u}, t)$ can be expressed as follows:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \end{cases}$$

Solving this vectorial differential equation, one obtains the “explicit” description of the linear dynamic system, i.e. the “explicit” form of the *state transition function* is obtained:

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^t \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau$$

where t_0 is the initial time instant, $\Phi(t, t_0)\mathbf{x}(t_0)$ is the *free evolution*, the term $\int_{t_0}^t \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau$ is the *forced evolution* and function $\Phi(t, t_0)$ is the state transition matrix of the system. Matrix $\Phi(t, t_0)$ can be computed solving the following matrix differential equation:

$$\frac{d}{dt}\Phi(t, t_0) = \mathbf{A}(t)\Phi(t, t_0)$$

with initial condition: $\Phi(t_0, t_0) = \mathbf{I}$.

Note: one can easily prove that the i -th column of matrix $\Phi(t, t_0)$ is the free evolution of the system obtained starting from the initial condition “ \mathbf{e}_i ”, that is the i -th column of the identity matrix \mathbf{I}_n .

The “implicit” form of continuous-time, linear, time-invariant systems is the following:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C} \mathbf{x}(t) + \mathbf{D} \mathbf{u}(t) \end{cases}$$

In this case, la *state transition matrix* $\Phi(t, t_0)$ is function of the difference $t - t_0$: $\Phi(t, t_0) = \Phi(t - t_0)$. Without losing generality one can set $t_0 = 0$, i.e. the *state transition matrix* $\Phi(t)$ is the solution of the following matrix differential equation:

$$\frac{d}{dt} \Phi(t) = \mathbf{A} \Phi(t)$$

with initial condition $\Phi(0) = \mathbf{I}$. The solution of this differential equation has the following form:

$$\Phi(t) = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \dots + \mathbf{A}^n \frac{t^n}{n!} + \dots \triangleq \mathbf{e}^{\mathbf{A}t}$$

This property can be easily verified as follows:

$$\dot{\Phi}(t) = \mathbf{A} \left[\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \dots + \mathbf{A}^n \frac{t^n}{n!} + \dots \right] = \mathbf{A} \Phi(t)$$

In the following the symbol $e^{\mathbf{A}t}$, or equivalently $\exp(\mathbf{A}t)$, will be use to denote the following *matrix exponential function*:

$$\mathbf{e}^{\mathbf{A}t} \triangleq \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!}$$

In this case the *state transition function* takes the following form:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau$$

When $t_0 = 0$ it is $\Phi(t) = \mathbf{e}^{\mathbf{A}t}$ and therefore one obtains:

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau$$

Example. Consider the following linear, continuous-time, time-variant system

$$\dot{x}(t) = a(t)x(t) + b(t)u(t)$$

and solve the differential equation starting from the initial condition $x(t_0) = x_0$.

- In the time-variant case, the state transition matrix $\Phi(t, t_0)$ is obtained solving the following homogeneous differential equation

$$\dot{\Phi}(t, t_0) = a(t)\Phi(t, t_0)$$

starting from the initial condition $\Phi(t_0, t_0) = 1$. Using the following transformations

$$\begin{aligned} \frac{\dot{\Phi}(t, t_0)}{\Phi(t, t_0)} = a(t) &\quad \rightarrow \quad \frac{d \ln \Phi(t, t_0)}{dt} = a(t) \quad \rightarrow \\ &\rightarrow \quad \ln \Phi(t, t_0) = \int_{t_0}^t a(\rho) d\rho \end{aligned}$$

one obtains

$$\Phi(t, t_0) = e^{\int_{t_0}^t a(\rho) d\rho}$$

The solution of the differential equation can be expressed as follows:

$$x(t) = e^{\int_{t_0}^t a(\rho) d\rho} x_0 + \int_{t_0}^t e^{\int_{\tau}^t a(\rho) d\rho} b(\tau) u(\tau) d\tau$$

- If the parameter $a(t)$ is constant, $a(t) = a$, the state transition matrix simplifies as follows

$$\Phi(t, t_0) = e^{a(t-t_0)}$$

and therefore the solution of the differential equation is

$$x(t) = e^{a(t-t_0)} x_0 + \int_{t_0}^t e^{a(t-\tau)} b(\tau) u(\tau) d\tau$$

If the input signal $u(t)$ and the parameter $b(t)$ are constant, i.e. $u(t) = u_0$ and $b(t) = b$, the solution simplifies as follows

$$x(t) = e^{a(t-t_0)} x_0 + \left(\int_{t_0}^t e^{a(t-\tau)} d\tau \right) b u_0$$

The term enclosed in brackets can be transformed as follows

$$\int_{t_0}^t e^{a(t-\tau)} d\tau = \frac{1}{-a} \left[e^{a(t-\tau)} \right]_{t_0}^t = \frac{e^{a(t-t_0)} - 1}{a}$$

and the solution of the differential equation is

$$x(t) = e^{a(t-t_0)} x_0 + \frac{e^{a(t-t_0)} - 1}{a} b u_0$$

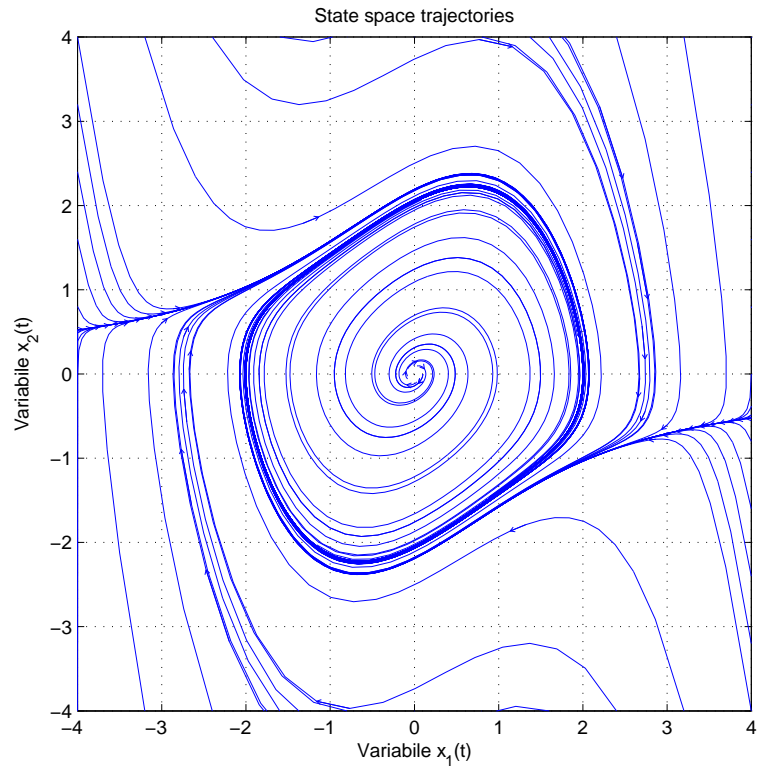
Example: Let us consider the following nonlinear, continuous-time, time-invariant system, known as Van Der Pol oscillator:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \beta(1 - x_1^2)x_2 \end{cases}$$

⇕

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

The state space trajectories of the system when $\beta = 0.5$ shows the presence of a limit cycle.



Matlab file "Van_Der_Pol.m":

```
-----
function Van_Der_Pol
% Van_Der_Pol oscillator:      x1 =x2
%                               x2 =-x1+beta*(1-x1^2)*x2
figure(1); clf
V=[[-1 1] [-1 1]]*4;          % Plot window
In_Con=inicond(V,[10,10]);    % Outer initial conditions
Circle=0.1*exp(1i*2*pi*(0:0.2:1)'); % Inner initial conditions
In_Con=[In_Con; real(Circle) imag(Circle)]; % Initial conditions
Tspan=(0:0.005:1)*20;        % Simulation time instant
fr=10; dx=0.06; dy=dx;      % Arrow position and arrow width
for jj=1:size(In_Con,1)
    [~,x]=ode23(@Van_Der_Pol_ode,Tspan,In_Con(jj,:)); % ODE simulation
    plot(x(:,1),x(:,2)); hold on % Plot
    freccia(x(fr,1),x(fr,2),x(fr+1,1),x(fr+1,2),dx,dy) % Draw the arrows
end
grid on; axis(V) % Grid and axis
axis square % Plot with a quered shape
xlabel('Variabile x_1(t)') % Label along axis x
ylabel('Variabile x_2(t)') % Label along axis y
title(['State Space trajectories']) % Title
print -depsc Van_Der_Pol.eps % Print the figure
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function dx=Van_Der_Pol_ode(t,x)
dx(1,1)=x(2);
dx(2,1)=-x(1)+0.5*(1-x(1)^2)*x(2);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function X0=inicond(V, dxy)
X0=zeros(2*(dxy(1)+dxy(2)),2);
for ii= 1:dxy(2); X0(ii,:)= [V(1) V(3)+(ii-1)*(V(4)-V(3))/dxy(2)]; end
for ii= 1:dxy(1); X0(dxy(2)+ii,:)= [V(1)+(ii-1)*(V(2)-V(1))/dxy(1) V(4)]; end
for ii= 1:dxy(2); X0(dxy(1)+dxy(2)+ii,:)= [V(2) V(4)-(ii-1)*(V(4)-V(3))/dxy(2)]; end
for ii= 1:dxy(1); X0(dxy(1)+2*dxy(2)+ii,:)= [V(2)-(ii-1)*(V(2)-V(1))/dxy(1) V(3)]; end
-----
```

Example: Let us consider the following second order linear system:

$$G(s) = \frac{100}{s^2 + s + 25} \Leftrightarrow \begin{cases} \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{b} u \\ y = \mathbf{c} \mathbf{x} + d u \end{cases}$$

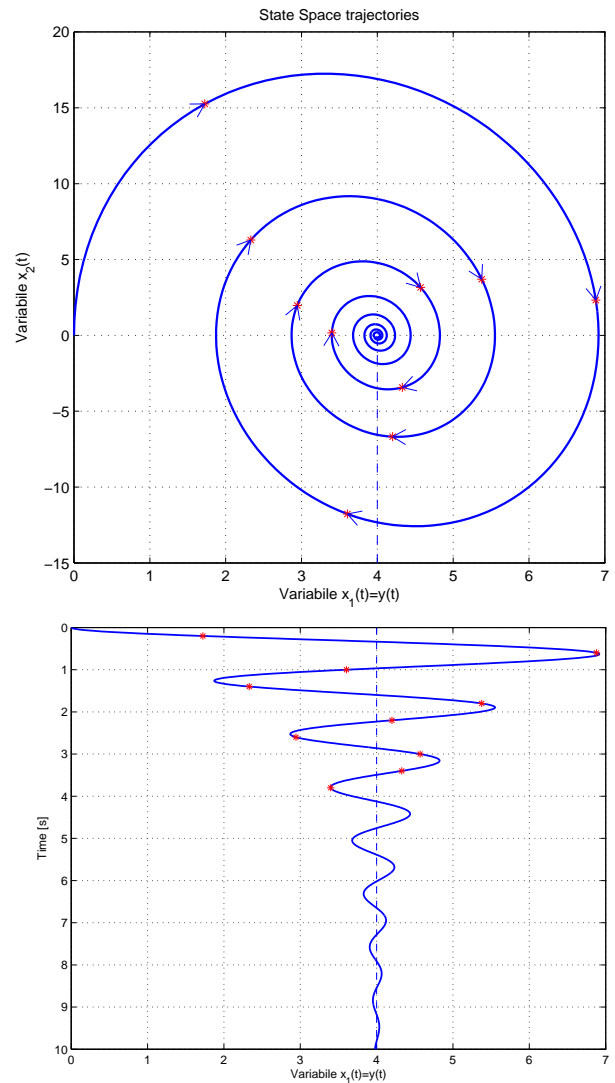
where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -25 & -1 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}, \quad \mathbf{c} = [1 \ 0], \quad d = 0.$$

The state space trajectory of the step response of system $G(s)$ shows that the final state is $\mathbf{x}_f = [y_\infty, 0]$, where $y_\infty = 4$.

The real trajectory has a spiral stable behavior in the vicinity of the final state \mathbf{x}_f and the output variable is the projection of this trajectory along the x_1 -axis.



Matlab file "Second_Order_Step_Trajectory.m":

```
-----
function Second_Order_Step_Trajectory
A=[0 1; -25 -1]; b=[0; 100]; c=[1 0]; d=0;
Gs=ss(A,b,c,d);
[yt,t,x]=step(Gs,0:0.01:10);
figure(1); clf
plot(x(:,1),x(:,2),'Linewidth',1.5); % Plot
hold on; axis square; V=axis; grid on;
dx=(V(2)-V(1))/35; dy=(V(4)-V(3))/35; % Arrow width
plot([4 4],[-15 0],'--')
for fr=20:40:400
    plot(x(fr+1,1),x(fr+1,2),'r*') % Plot the arrow points
    freccia(x(fr,1),x(fr,2),x(fr+1,1),x(fr+1,2),dx,dy) % Plot the arrows
end
xlabel('Variabile x_1(t)=y(t)') % Label along axis x
ylabel('Variabile x_2(t)') % Label along axis y
title('State Space trajectories') % Title
figure(2); clf;
plot(yt,t,'Linewidth',1.5); hold on;
plot([4 4],[0 10],'--')
set(get(2,'Children'),'Ydir','reverse'); grid on;
for fr=20:40:400
    plot(yt(fr+1),t(fr+1),'r*') % Draw the arrow points
end
ylabel('Time [s]'); xlabel('Variabile x_1(t)=y(t)')
-----
```

Linear, discrete-time dynamic systems

A non-linear, discrete-time, finite-dimensional, time-variant dynamic system can always be described in the following “implicit” form:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k), k) \\ \mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k), k) \end{cases}$$

If the dynamic system is “linear” and time-variant, the *state function* $\mathbf{f}(\cdot)$ and the *output function* $\mathbf{g}(\cdot)$ take the following form:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k) \end{cases}$$

Solving this vectorial difference equation one obtains the following “explicit” description of the considered dynamic system:

$$\mathbf{x}(k) = \Phi(k, h)\mathbf{x}(h) + \sum_{j=h}^{k-1} \Phi(k, j+1)\mathbf{B}(j)\mathbf{u}(j)$$

where h is initial time, $\Phi(k, h)\mathbf{x}(h)$ is the *free evolution*, the term $\sum_{j=h}^{k-1} \Phi(k, j+1)\mathbf{B}(j)\mathbf{u}(j)$ is the *forced evolution* and $\Phi(k, h)$ is the “*state transition matrix*” defined as follows:

$$\Phi(k, h) = \begin{cases} \mathbf{A}(k-1) \dots \mathbf{A}(h+1)\mathbf{A}(h) & \text{se } k > h \\ \mathbf{I} \text{ (Matrice identità)} & \text{se } k = h \end{cases}$$

If the discrete-time system is *time-invariant* the matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} are constant, i.e. they are not function of the time t :

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A} \mathbf{x}(k) + \mathbf{B} \mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C} \mathbf{x}(k) + \mathbf{D} \mathbf{u}(k) \end{cases}$$

In this case it is

$$\mathbf{A}(h) = \mathbf{A}(h+1) = \dots = \mathbf{A}(k-1) = \mathbf{A}$$

and the *state transition matrix* $\Phi(k, h)$ simplifies as follows:

$$\Phi(k, h) = \mathbf{A}(k-1)\mathbf{A}(k-2)\dots\mathbf{A}(h) = \mathbf{A}^{k-h}$$

Choosing $h = 0$, the *state transition function* is:

$$\mathbf{x}(k) = \mathbf{A}^k \mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{(k-j-1)} \mathbf{B} \mathbf{u}(j)$$

The forced evolution can also be rewritten in the following matrix form:

$$\sum_{j=0}^{k-1} \mathbf{A}^{(k-j-1)} \mathbf{B} \mathbf{u}(j) = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{k-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}(k-1) \\ \mathbf{u}(k-2) \\ \mathbf{u}(k-3) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}$$

Example. Consider the following linear, discrete-time, time-variant system

$$x(k+1) = \left(\alpha + \frac{\beta}{k} \right) x(k) + b(k)u(k)$$

defined for $k \geq 1$. Solve this linear, time-variant, difference equation with initial condition $x(1) = x_1$. In the general time-variant case, the state transition matrix is

$$\Phi(k, h) = \begin{cases} \left(\alpha + \frac{\beta}{k-1} \right) \left(\alpha + \frac{\beta}{k-2} \right) \dots \left(\alpha + \frac{\beta}{h} \right) & \text{per } k > h \\ 1 & \text{per } k = h \end{cases}$$

The solution of the difference equation is:

$$x(k) = \Phi(k, 1)x_1 + \sum_{i=1}^{k-1} \Phi(k, i+1)b(i)u(i)$$

If $\beta = 0$, the state transition matrix simplifies as follows

$$\Phi(k, h) = \alpha^{k-h}$$

and the solution of the difference equation is

$$x(k) = \alpha^{k-1}x_1 + \sum_{i=1}^{k-1} \alpha^{k-i-1}b(i)u(i)$$

If the input signal $u(k)$ and the parameter $b(k)$ are constant, i.e. $u(k) = u_0$ and $b(k) = b$, the solution simplifies as follows

$$x(k) = \alpha^{k-1}x_1 + \left(\sum_{i=1}^{k-1} \alpha^{k-i-1} \right) b u_0$$

The term enclosed in brackets can be expressed as follows

$$\sum_{i=1}^{k-1} \alpha^{k-i-1} = \sum_{i=0}^{k-2} \alpha^i = \frac{1 - \alpha^{k-1}}{1 - \alpha}$$

and the solution transforms as follows

$$x(k) = \alpha^{k-1}x_1 + \frac{1 - \alpha^{k-1}}{1 - \alpha} b u_0$$

The step response of the following linear, discrete-time system

$$x(k+1) = \alpha x(k) + b u(k)$$

can also be obtained using the \mathcal{Z} -transform technique:

$$z[X(z) - x(1)] = \alpha X(z) + b U(z)$$

Using $x(1) = x_1$ and solving with respect to $X(z)$ one obtains:

$$X(z) = \frac{z}{z - \alpha} x_1 + \frac{b}{z - \alpha} U(z)$$

If the input $u(k)$ is a unit step:

$$U(z) = \frac{u_0 z}{z - 1}$$

the solution is

$$\begin{aligned} X(z) &= \frac{z}{z - \alpha} x_1 + \frac{b u_0 z}{(z - \alpha)(z - 1)} \\ &= \frac{z}{z - \alpha} x_1 + \frac{b u_0 z}{(1 - \alpha)} \left[\frac{1}{z - 1} - \frac{1}{z - \alpha} \right] \end{aligned}$$

The inverse \mathcal{Z} -transform of function $X(z)$ (with an additional one time delay) is:

$$x(k) = \alpha^{k-1}x_1 + b u_0 \frac{1 - \alpha^{k-1}}{1 - \alpha}$$