Trajectories in the state space

To show the typical behaviors of the state space trajectories as a function of the position of the system poles, we will refer to the following example, a dynamic system with a mass, a spring and a dumper.

The differential equation describing the dynamics of the system composed by a mass M, a spring K, a dumper b and an external force F is the following :

$$F = M\ddot{x} + b\dot{x} + Kx \qquad \rightarrow \qquad \ddot{x} = -\frac{K}{M}x - \frac{b}{M}\dot{x} + F$$

An equivalent state space dynamic description of the system is the following:

$$(1) \quad \left\{ \begin{array}{rcl} \dot{\mathbf{x}}(t) & = & \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{b}{M} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{array} \right.$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

The dynamic behaviors of the system is a function of the numerical values of the parameters. We will consider three different cases.

Case I. Setting $M = 1$, $K = 2$ and $b = 3$ one obtains the following system:

$$(2) \quad \left\{ \begin{array}{rcl} \dot{\mathbf{x}}(t) & = & \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{array} \right.$$

The characteristic polynomial of matrix $\mathbf{A}$ is:

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{bmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

The eigenvalues of matrix $\mathbf{A}$ are:

$$\lambda_1 = -1, \qquad \lambda_2 = -2$$

The two distinct eigenvalues λ_1 and λ_2 generate two linearly independent eigenvalues $\mathbf{v}_1$ and $\mathbf{v}_2$ which can be determined solving the following linear homogeneous systems

$$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{v}_1=\mathbf{o}, \qquad\qquad (\mathbf{A}-\lambda_2\mathbf{I})\mathbf{v}_2=\mathbf{o}$$

that is

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\mathbf{v}_1=\mathbf{o} \qquad \rightarrow \qquad \mathbf{v}_1=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\mathbf{v}_2=\mathbf{o} \qquad \rightarrow \qquad \mathbf{v}_2=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Let us consider the state space transformation $\mathbf{x}=\mathbf{T}\,\overline{\mathbf{x}}$ where

$$\mathbf{T}=[\mathbf{v}_1\ \mathbf{v}_2]=\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, \qquad\qquad \mathbf{T}^{-1}=\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}.$$

The new system matrix $\overline{\mathbf{A}}$ has a diagonal form with the eigenvalues located on the main diagonal:

$$\overline{\mathbf{A}}=\mathbf{T}^{-1}\mathbf{A}\mathbf{T}=\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

The exponential of the diagonal matrix $\overline{\mathbf{A}}t$ is:

$$e^{\overline{\mathbf{A}}t}=\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

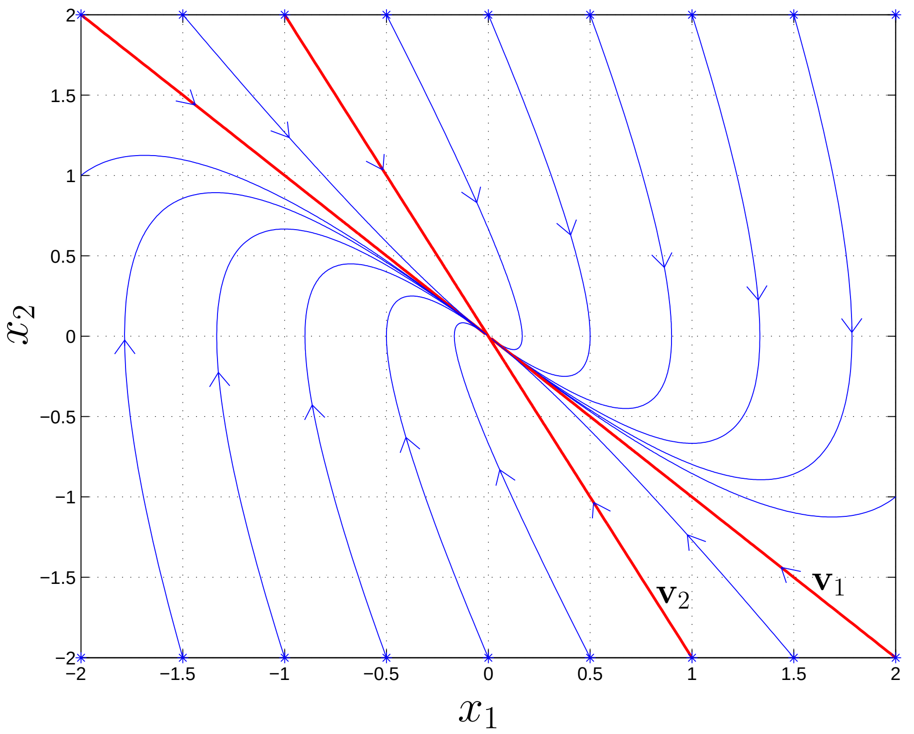
The free evolution of system (2) starting from the initial condition $\mathbf{x}_0$ is:

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0=e^{(\mathbf{T}\overline{\mathbf{A}}\mathbf{T}^{-1})t}\,\mathbf{x}_0=\mathbf{T}e^{\overline{\mathbf{A}}t}\mathbf{T}^{-1}\mathbf{x}_0\\ &= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\mathbf{x}_0\\ &= \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}\mathbf{x}_0\end{aligned}$$

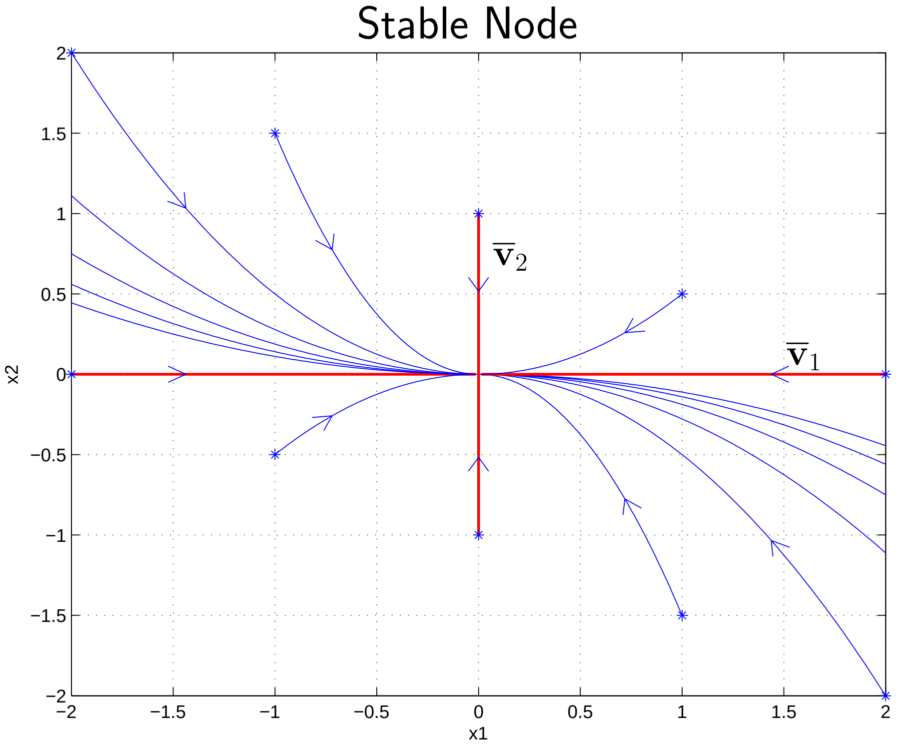
Computation of $e^{\mathbf{A}t}$ in Matlab:

```
A=[ 0  1; ...                     +-                                             -+
   -2 -3];                        |   2 exp(-t) - exp(-2 t),   exp(-t) - exp(-2 t)  |
syms t;               -->         |                                                 |
Eat=expm(A*t);                    |  2 exp(-2 t) - 2 exp(-t), 2 exp(-2 t) - exp(-t)  |
pretty(Eat)                       +-                                             -+
```

The behavior of the state space trajectories $\mathbf{x}(t)$ in the "original" space is the following:



The two red trajectories correspond to the two subspaces U_{λ_1} and U_{λ_2} associated to the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The same trajectories in the transformed space $\overline{\mathbf{x}}(t)$ have the following behavior:



Trajectories of this type are referred to as: "**Stable Node**".

Case II. Setting $M = 1$, $K = 1$ and $b = 2$ the system becomes:

$$(3) \quad \left\{ \begin{array}{rcl} \dot{\mathbf{x}}(t) &=& \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y &=& \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{array} \right.$$

The characteristic polynomial of matrix $\mathbf{A}$ is: $\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$. The matrix $\mathbf{A}$ has only one eigenvalue, $\lambda_{1,2} = -1$, with algebraic multiplicity 2. The corresponding eigenvector $\mathbf{v}_1$ can be determined solving the following homogeneous system:

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{v}_1 = \mathbf{o} \qquad \rightarrow \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A second order generalized eigenvector $\mathbf{v}_2$ is then determined solving the following linear system:

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1 \quad \leftrightarrow \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

let us now consider the following state space transformation $\mathbf{x} = \mathbf{T}\overline{\mathbf{x}}$ where:

$$\mathbf{T} = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The new state matrix $\overline{\mathbf{A}}$ has the following form:

$$\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \rightarrow \qquad e^{\overline{\mathbf{A}}t} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

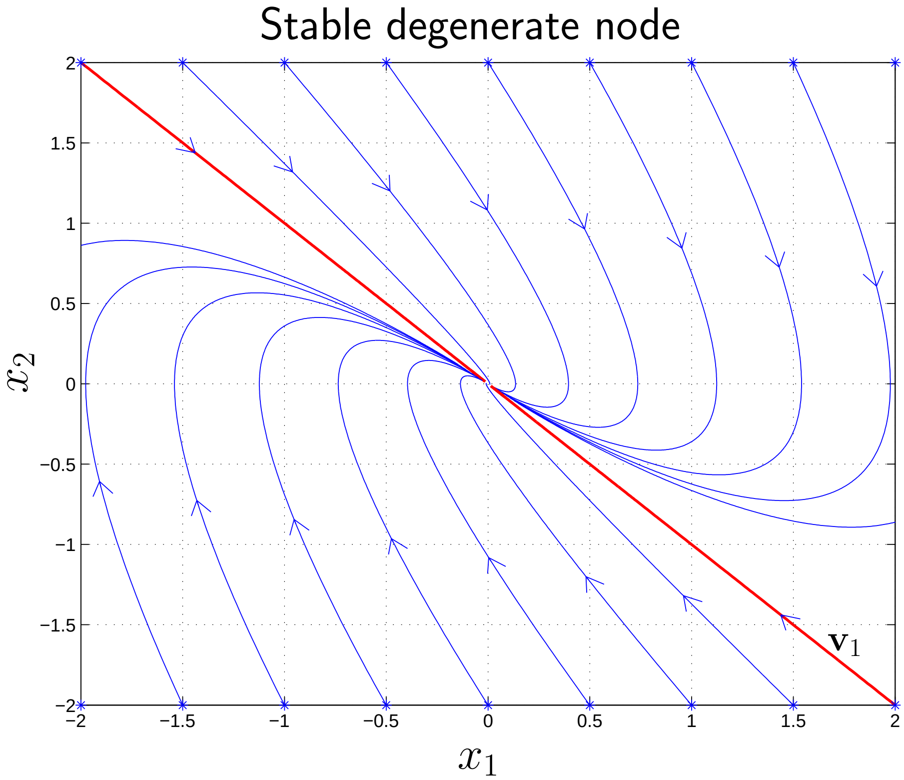
The free evolution of system (3) starting from the initial condition $\mathbf{x}_0$ is:

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 = e^{\mathbf{T}\overline{\mathbf{A}}\mathbf{T}^{-1}t}\mathbf{x}_0 = \mathbf{T}e^{\overline{\mathbf{A}}t}\mathbf{T}^{-1}\mathbf{x}_0 \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}_0 \\ &= \begin{bmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix} \mathbf{x}_0 \end{aligned}$$

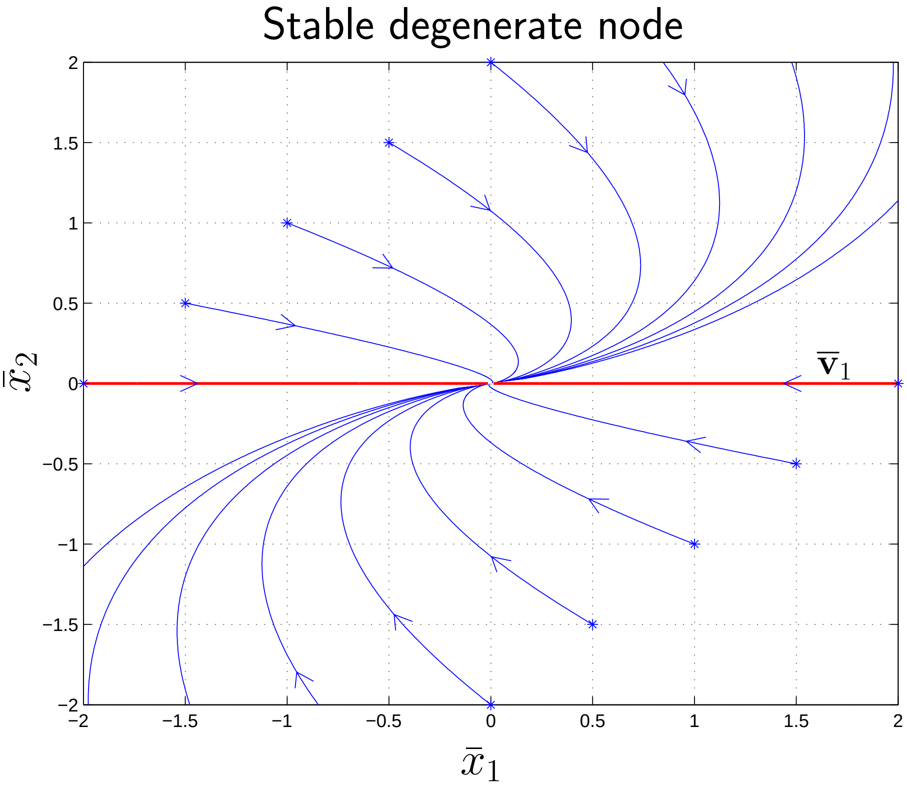
Computation of $e^{\mathbf{A}t}$ in Matlab:

```
A= [0  1; ...                             +-                                    -+
   -1 -2];                                |  exp(-t) (t + 1),      t exp(-t)     |
syms t;                       -->         |                                      |
Eat=simplify(expm(A*t));                  |     -t exp(-t),    -exp(-t) (t - 1)  |
pretty(Eat)                               +-                                    -+
```

The trajectories $\mathbf{x}(t)$ in the state space $(x_1,\ x_2)$ is the following:



The red trajectory corresponds to the eigenvector $\mathbf{v}_1$. The behavior of trajectories $\overline{\mathbf{x}}(t)$ in the transformed space $(\bar{x}_1,\ \bar{x}_2)$ is the following:



Trajectories of this type are called: "**Stable degenerate node**".

Case III. Choosing $M = 1$, $K = 2$ and $b = 2$ one obtains the following system:

$$(4)\quad \left\{ \begin{array}{rcl} \dot{\mathbf{x}}(t) & = & \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{array} \right.$$

The characteristic polynomial is now the following:

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 2 \end{bmatrix} = \lambda^2 + 2\lambda + 2 = (\lambda + 1)^2 + 1$$

Matrix $\mathbf{A}$ has two complex conjugate eigenvalues $\lambda_{1,2} = -1 \pm j$ to which corresponds two complex conjugate eigenvectors $\mathbf{v}_{1,2}$. The eigenvector $\mathbf{v}_1$ can be determined solving the following homogeneous system:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \begin{bmatrix} 1 - j & 1 \\ -2 & -1 - j \end{bmatrix} \mathbf{v}_1 = \mathbf{o} \qquad \rightarrow \qquad \mathbf{v}_1 = \begin{bmatrix} 1 + j \\ -2 \end{bmatrix}$$

Using a state space transformation $\mathbf{x} = \overline{\mathbf{T}}\overline{\mathbf{x}}$ based on the complex conjugate eigenvectors $\mathbf{v}_{1,2}$, it is possible to "diagonalize" the matrix $\mathbf{A}$:

$$\overline{\mathbf{T}} = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} 1 + j & 1 - j \\ -2 & 2 \end{bmatrix}, \qquad \overline{\mathbf{A}} = \overline{\mathbf{T}}^{-1}\mathbf{A}\overline{\mathbf{T}} = \begin{bmatrix} -1 + j & 0 \\ 0 & -1 - j \end{bmatrix}$$

The obtained transformed system is complex and difficult to use. In these cases it is preferable to use the "real Jordan form".

In this case the columns of the transformation matrix $\mathbf{T}$ are equal to the real part $\mathbf{w}_1$ and the imaginary part $\mathbf{w}_2$ of the previously calculated eigenvector $\mathbf{v}_1$:

$$\mathbf{v}_1 = \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{\mathbf{w}_1} + j \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{w}_2} \qquad \mathbf{T} = [\mathbf{w}_1\ \mathbf{w}_2] = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}, \qquad \mathbf{T}^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$$

The obtained transformed matrix $\overline{\mathbf{A}}$ has the following structure:

$$\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

which is a particular value of the following general case:

$$\overline{\mathbf{A}} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

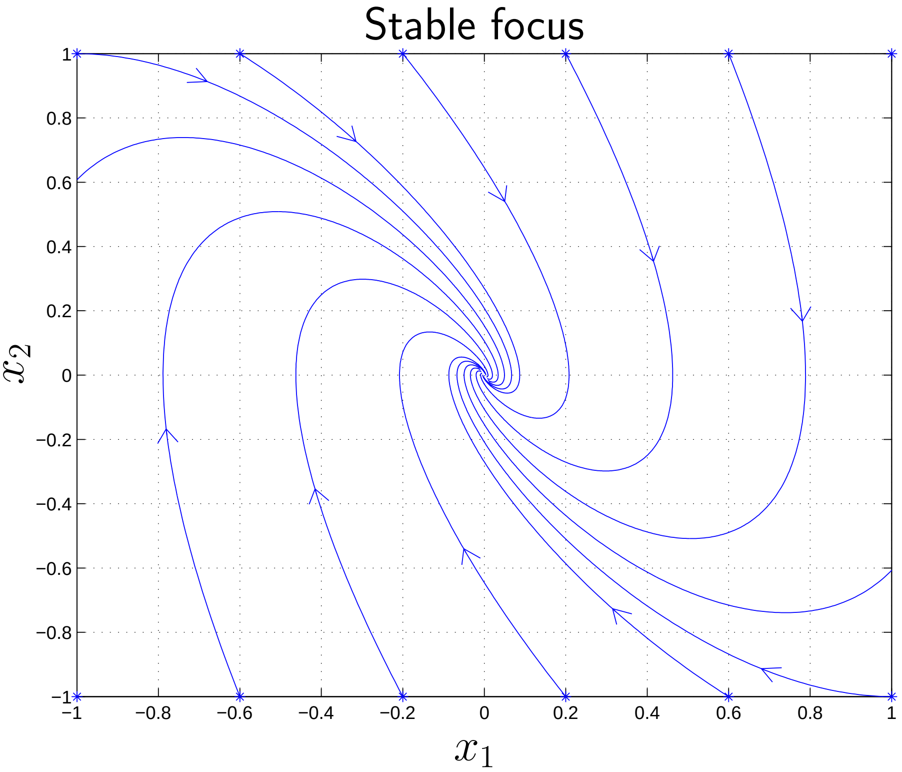
The free evolution of system (4) starting from the initial condition $\mathbf{x}_0$ is:

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 = e^{\mathbf{T}\bar{\mathbf{A}}\mathbf{T}^{-1}t}\mathbf{x}_0 = \mathbf{T}e^{\bar{\mathbf{A}}t}\mathbf{T}^{-1}\mathbf{x}_0\\ &= \mathbf{T}\begin{bmatrix} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t\\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t\end{bmatrix}\mathbf{T}^{-1}\mathbf{x}_0\end{aligned}$$

In this case it is $\lambda_1 = \sigma + j\omega = -1 + j$ and the following solution is obtained:

$$\begin{aligned}\mathbf{x}(t) &= \begin{bmatrix} 1 & 1\\ -2 & 0\end{bmatrix}\begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t\\ -e^{-t}\sin t & e^{-t}\cos t\end{bmatrix}\begin{bmatrix} 0 & -\frac{1}{2}\\ 1 & \frac{1}{2}\end{bmatrix}\mathbf{x}_0\\ &= \begin{bmatrix} e^{-t}(\cos t + \sin t) & e^{-t}\sin t\\ -2e^{-t}\sin t & e^{-t}(\cos t - \sin t)\end{bmatrix}\mathbf{x}_0\end{aligned}$$

The behavior of trajectories $\mathbf{x}(t)$ in the state space $(x_1,\ x_2)$ is the following:

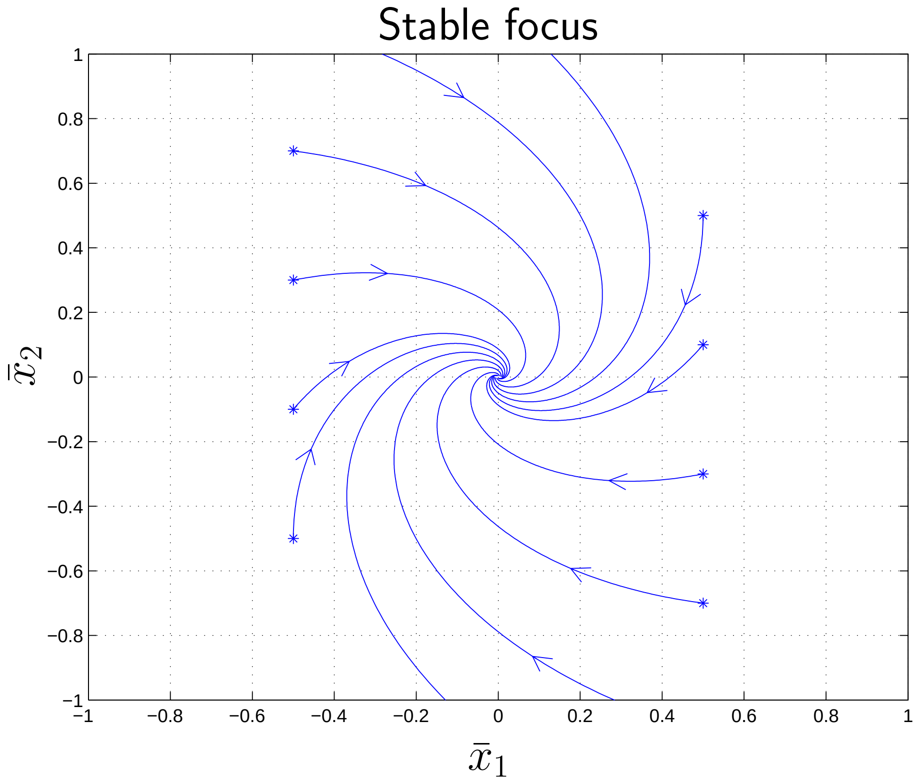


The trajectories tend to the origin with a spiral behavior. Trajectories of this type are referred to as: "**Stable focus**".

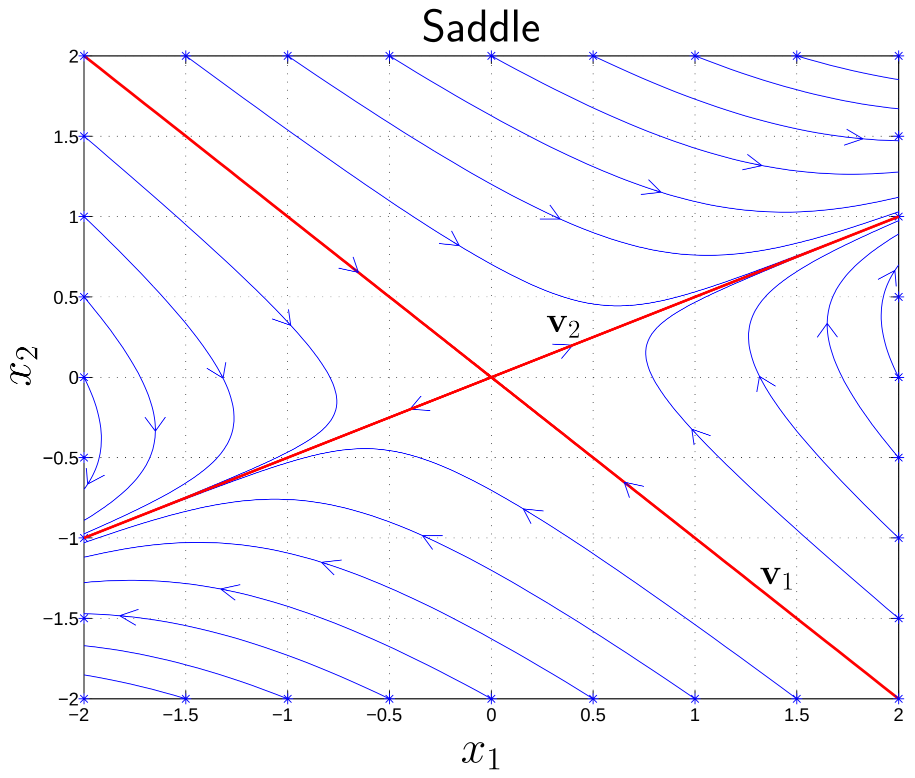
Computation of $e^{\mathbf{A}t}$ in Matlab:

```
A= [0  1; ...                          +-                                                     -+
   -2 -2];                             |  exp(-t) (cos(t) + sin(t)),       exp(-t) sin(t)      |
syms t;                     -->        |                                                       |
Eat=simplify(expm(A*t));               |      -2 exp(-t) sin(t),     exp(-t) (cos(t) - sin(t)) |
pretty(Eat)                            +-                                                     -+
```

The behavior of trajectories $\overline{\mathbf{x}}(t)$ in the transformed state space is the following:


Stable focus

The trajectories of a second order system having a negative eigenvalue $\lambda_1 < 0$ and a positive eigenvalue $\lambda_2 > 0$ are called "**Saddle**" and have the following behavior:

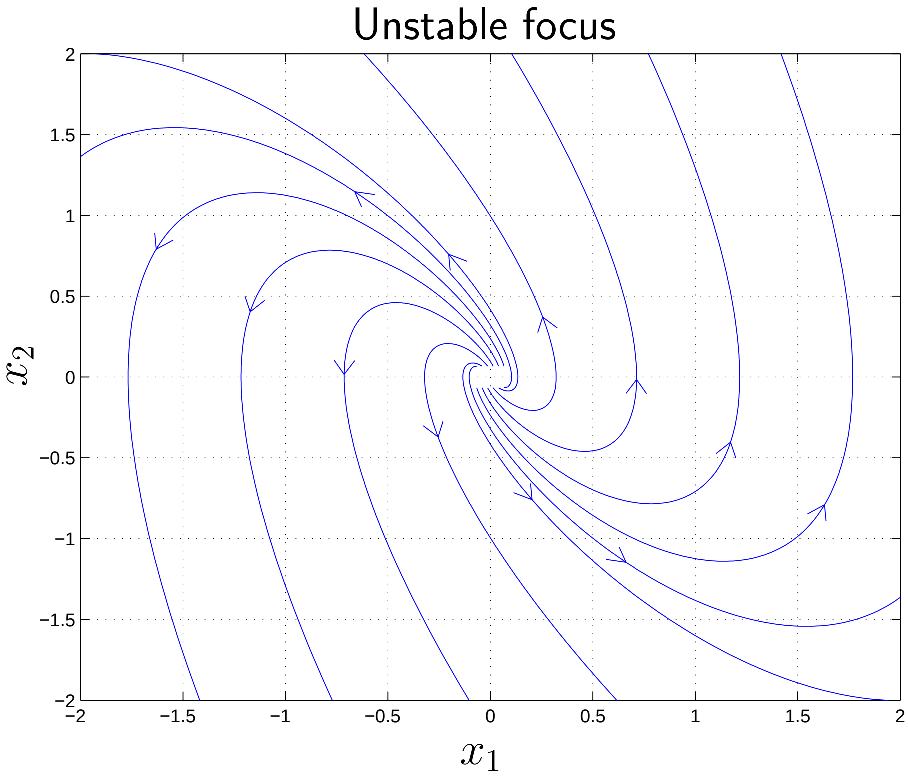

Saddle

The two red trajectories correspond to the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ associated to the two eigenvalues λ_1 and λ_2.

- Typical behavior of trajectories belonging to an "**Unstable focus**":

$$\overline{\mathbf{A}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

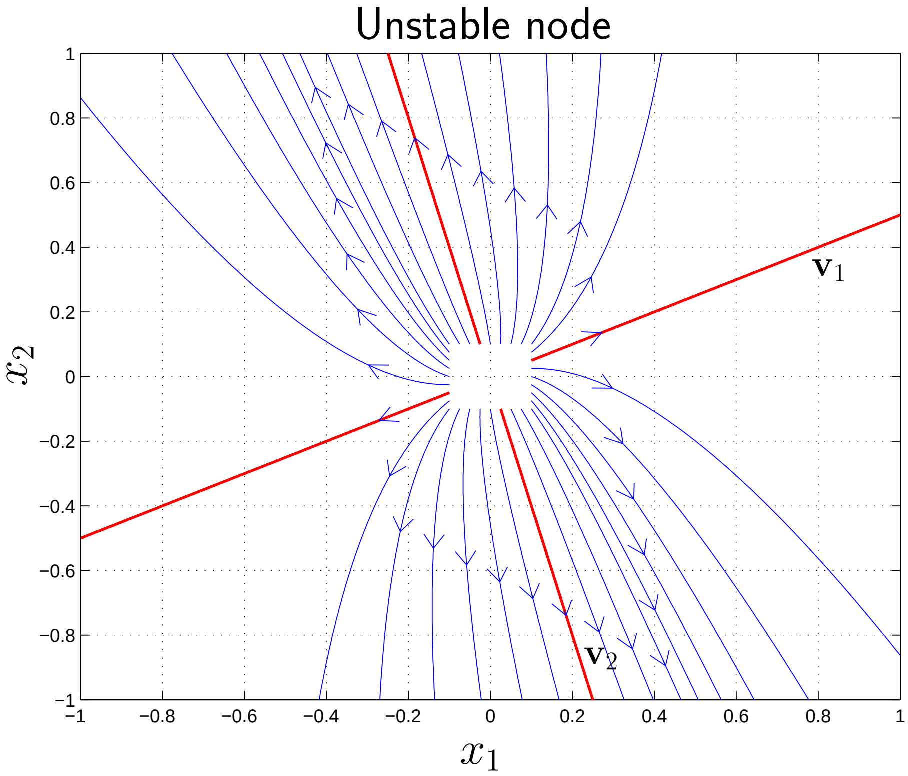
$$\lambda_{1,2} = 1 \pm j$$



- Typical behavior of trajectories belonging to an "**Unstable node**":

$$\overline{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = 2$$



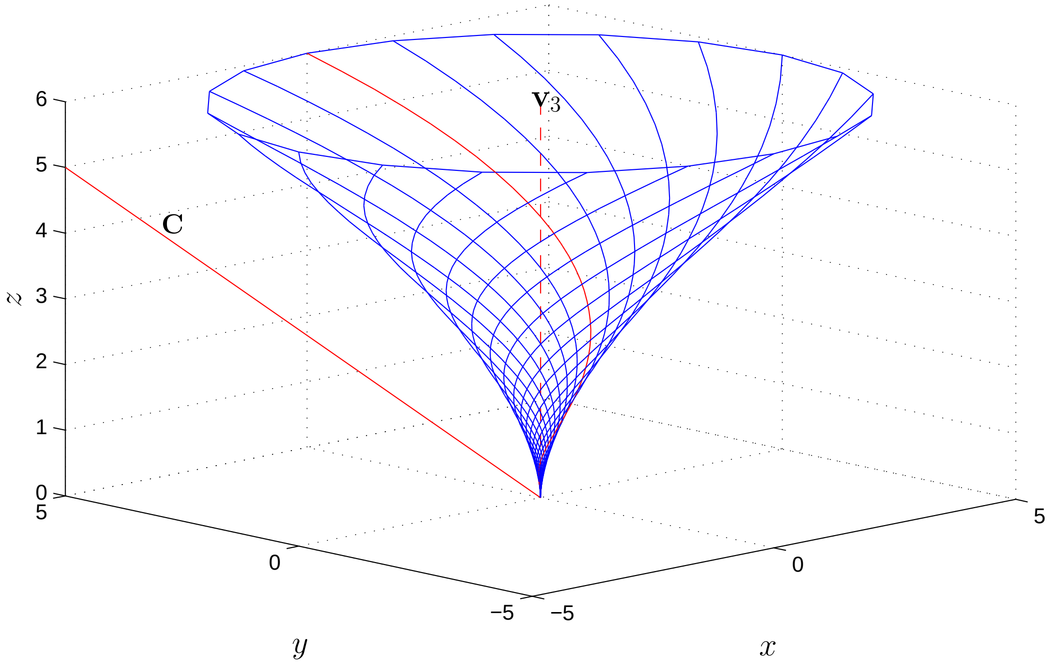
In this case the dominant eigenvalue is $\lambda_2 = 2$ and therefore all the trajectories tend to become parallel to the eigenvector $\mathbf{v}_2$.

Free trajectories in a three dimensional space.

The free trajectories of a linear system having a real eigenvalue $\lambda_3 = -3$ which is dominant with respect to the other two complex eigenvalues $\lambda_{1,2} = -5 \pm 5\,j$:

$$\overline{\mathbf{A}} = \begin{bmatrix} -5 & 5 & 0 \\ -5 & -5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

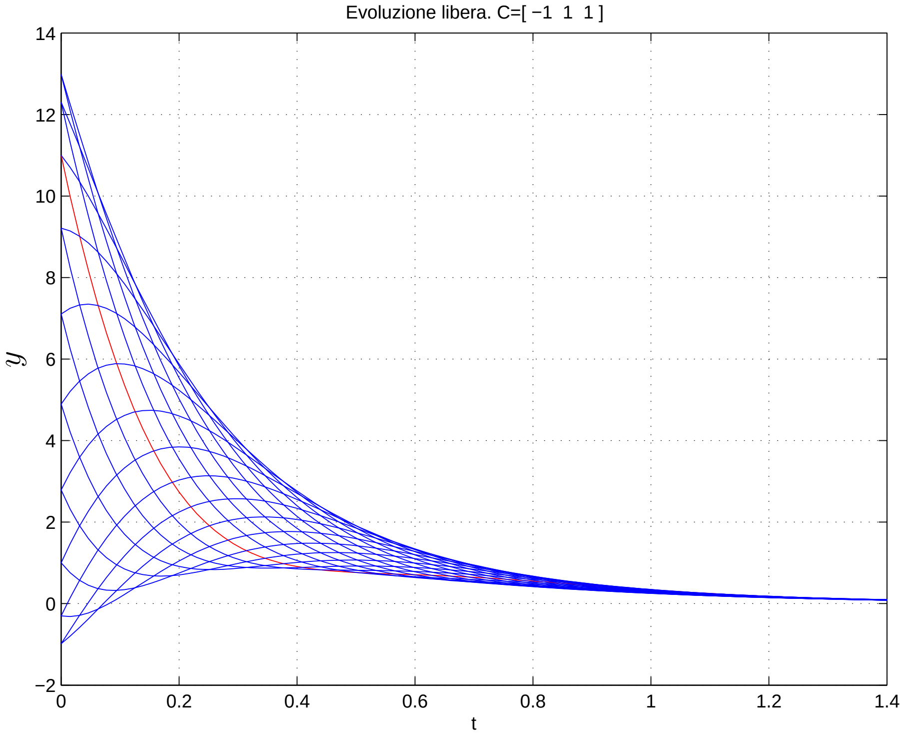
$$\begin{aligned} \lambda_{1,2} &= -5 \pm 5j \\ \lambda_3 &= -3 \end{aligned}$$



All the trajectories tend towards the dominant eigenvector $\mathbf{v}_3 = [0\ 0\ 1]^T$. The free evolution of the "output" signal $y(t)$ is:

$$y(t) = \mathbf{c}\,\mathbf{x}(t)$$

$$\mathbf{c} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$$

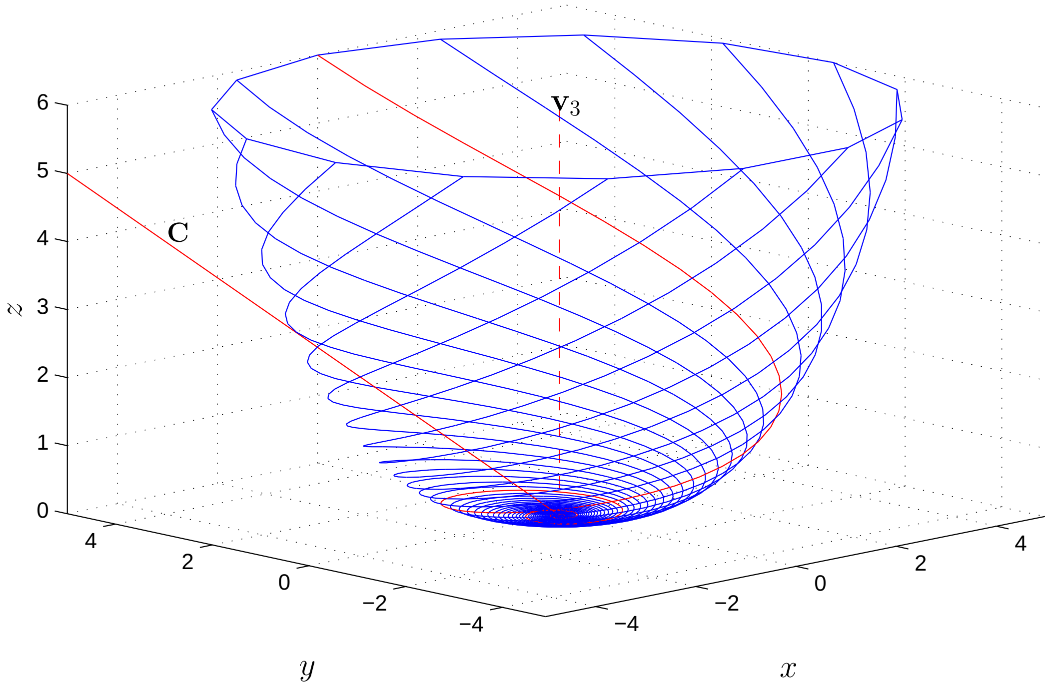


All the free evolutions of the "output" signal $y(t)$ have an "aperiodic" behavior which is determined by the dominant eigenvalue $\lambda_3 = -3$.

The free trajectories of a linear system having two complex eigenvalues $\lambda_{1,2} = -1 \pm 5\,j$ dominant with respect to the third eigenvalue $\lambda_3 = -3$ are:

$$\overline{\mathbf{A}} = \begin{bmatrix} -1 & 5 & 0 \\ -5 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

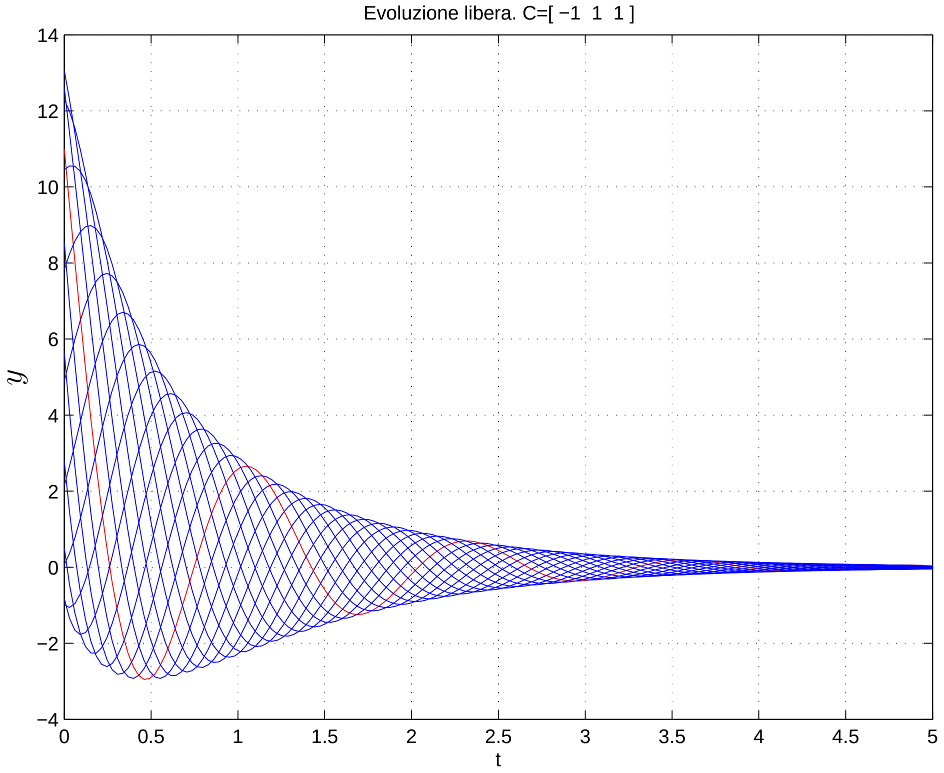
$$\begin{aligned} \lambda_{1,2} &= -1 \pm 5j \\ \lambda_3 &= -3 \end{aligned}$$



All the trajectories tend towards the plane $(x,\, y)$ determined by the two complex eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ associated to the eigenvalues $\lambda_{1,2} = -1 \pm 5\,j$. In this case the free evolutions of the output signal $y(t)$ are:

$$y(t) = \mathbf{c}\,\mathbf{x}(t)$$

$$\mathbf{c} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$$



All the free evolutions of the output signal $y(t)$ have a stable decreasing oscillatory behavior determined by the two complex eigenvalues $\lambda_{1,2} = -1 \pm 5\,j$. The state space trajectories have been projected along the $\mathbf{c}$ vector.