State estimators

- The static state feedback $\mathbf{u}(k) = \mathbf{K}\,\mathbf{x}(k)$ can be used only if the state vector $\mathbf{x}(k)$ is completely known. Usually, the only variables which are measured, using sensors, are the components of the output vector $\mathbf{y}(k)$.
- Since the state $\mathbf{x}(k)$ is unknown, one tries to obtain an estimation $\hat{\mathbf{x}}(k)$ of the state using a properly designed dynamic system which is called *state estimator*. Three different types of state estimators will be considered: 1) the open loop estimator; 2) the closed loop estimator; 3) the reduced order estimator.

1) The open loop state estimator

- The *open loop state estimator* can be easily obtained writing a "copy" of the given dynamic system. The dynamic equations of the open loop state (for both discrete and continuous-time cases) are the following:

$$\boxed{\hat{\mathbf{x}}(k+1) = \mathbf{A}\,\hat{\mathbf{x}}(k) + \mathbf{B}\,\mathbf{u}(k)}, \qquad \boxed{\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\,\hat{\mathbf{x}}(t) + \mathbf{B}\,\mathbf{u}(t)}$$

- *Property. The open loop state estimator can be used only if systems which are asymptotically stable.*

  Proof. Let us define as *estimation errors* (for both discrete and continuous-time cases) the following variables:

$$\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k), \qquad \mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$$

  Substituting one obtains that $\mathbf{e}(k+1) = \mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1) = \mathbf{A}[\mathbf{x}(k) - \hat{\mathbf{x}}(k)]$ e $\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}[\mathbf{x}(t) - \hat{\mathbf{x}}(t)]$, from which one obtains:

$$\mathbf{e}(k+1) = \mathbf{A}\mathbf{e}(k), \qquad \dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t)$$

  Let $\mathbf{e}(0) = \hat{\mathbf{x}}(0) - \mathbf{x}(0)$ be the initial estimation error. In the discrete and continuous-time cases, the dynamics of the estimation error is:

$$\mathbf{e}(k) = \mathbf{A}^k\mathbf{e}(0), \qquad \mathbf{e}(t) = e^{\mathbf{A}\,t}\mathbf{e}(0)$$

  The estimation error tends to zero only if the eigenvalues of matrix $\mathbf{A}$ are strictly stable.

- For stable systems, however, the velocity of the estimation error $\hat{\mathbf{x}}(k)$ towards zero $\mathbf{x}(k)$ cannot be modified: it is a function of the dominant eigenvalue of matrix $\mathbf{A}$.

2) Closed loop state estimator

- The *closed loop estimator* can be obtained from the open loop estimator adding a feedback control action $\mathbf{L}[\hat{\mathbf{y}}(k) - \mathbf{y}(k)]$ (for the discrete-time case) proportional to the output estimation error $\hat{\mathbf{y}}(k) - \mathbf{y}(k)$:

$$\hat{\mathbf{x}}(k+1) = \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) + \underbrace{\mathbf{L}[\hat{\mathbf{y}}(k) - \mathbf{y}(k)]}$$

- The equations of the closed loop estimator are the following:

$$\boxed{\hat{\mathbf{x}}(k+1) = (\mathbf{A} + \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(k) - \mathbf{L}\mathbf{y}(k) + \mathbf{B}\mathbf{u}(k)} \qquad \text{(discrete case)}$$

  For discrete systems the estimation error satisfies the following equation:

$$\mathbf{e}(k+1) = (\mathbf{A} + \mathbf{L}\mathbf{C})\mathbf{e}(k) \quad \rightarrow \quad \mathbf{e}(k) = (\mathbf{A} + \mathbf{L}\mathbf{C})^k \, \mathbf{e}(0)$$

  The estimation error $\mathbf{e}(k)$ tends to zero when $k \rightarrow \infty$ if and only if all the eigenvalues of matrix $\mathbf{A} + \mathbf{L}\mathbf{C}$ are asymptotically stable, that is when they have modulus less than 1.

  Matrix $\mathbf{L}$ is a free design parameter which can be used to locate arbitrarily the eigenvalues of matrix $\mathbf{A} + \mathbf{L}\mathbf{C}$.

- For continuous-time systems, the matrix equation of the closed loop estimator is the following:

$$\boxed{\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} + \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) - \mathbf{L}\mathbf{y}(t) + \mathbf{B}\mathbf{u}(t)} \qquad \text{(continuous case)}$$

  For continuous-time systems the estimation error $\mathbf{e}(t)$ satisfies the following equaiton:

$$\dot{\mathbf{e}}(t) = (\mathbf{A} + \mathbf{L}\mathbf{C})\mathbf{e}(t) \quad \rightarrow \quad \mathbf{e}(t) = e^{(\mathbf{A}+\mathbf{L}\mathbf{C})t} \, \mathbf{e}(0)$$

  The estimation error $\mathbf{e}(t)$ tends to zero when $t \rightarrow \infty$ is and only if all the eigenvalues of matrix $\mathbf{A} + \mathbf{L}\mathbf{C}$ are asymptotically stable, that is they have all a negative real part.

- Matrix $\mathbf{L}$ can be designed in a dual "manner" as follows: the dual "reachable" system $(\mathbf{A}^T, \mathbf{C}^T) = (\mathbf{A}_D, \mathbf{B}_D)$ is considered, and then matrix $\mathbf{L}^T = \mathbf{K}_D$ is designed to locate arbitrarily the eigenvalues of the reachable subsystem using the methods described in the previous sections.

- If the system is not completely observable, the eigenvalues of the unobservable subsystem (and the corresponding "modes" of the estimation error) cannot be modified using matrix $\mathbf{L}$.

- **An asymptotically stable estimator can be designed if and only if the unobservable part of the system is stable.**

- For observable systems $(\mathbf{A},\ \mathbf{C})$ with only one output $(p = 1)$, instead of using the "dual" design method, the eigenvalues of matrix $\mathbf{A} + \mathbf{LC}$ can be arbitrarily chosen using the two following 'dual" formulae.

  1) The dual formula that uses the *observability canonical form*:

$$\mathbf{L} = \mathbf{P}_c\mathbf{L}_c = \left\{ \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & 1 \\ \alpha_2 & \ldots & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \ldots \\ \mathbf{CA}^{n-1} \end{bmatrix} \right\}^{-1} \underbrace{\begin{bmatrix} \alpha_0 - d_0 \\ \alpha_1 - d_1 \\ \vdots \\ \alpha_{n-1} - d_{n-1} \end{bmatrix}}_{\mathbf{L}_c}$$

  where α_i are the coefficients of the characteristic polynomial of matrix $\mathbf{A}$:

$$\Delta_{\mathbf{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0$$

  and d_i are the coefficients of the freely chosen monic polynomial $p(\lambda)$:

$$p(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0$$

  2) The dual *Ackerman formula*:

$$\mathbf{L} = -p(\mathbf{A})\, \underbrace{(\mathcal{O}^-)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{q}} = -p(\mathbf{A})\mathbf{q}$$

  where $\mathbf{q}$ is the last column of the inverse of the observability matrix $\mathcal{O}^-$ and $p(\mathbf{A})$ is the matrix obtained from polynomial $p(\lambda)$ when the parameter λ is substituted by matrix $\mathbf{A}$.

Example. Let us consider the following continuous-time linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -2 & 1 & -1 \\ -2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}, \qquad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

where $\mathbf{x}(t)$ is the state vector, $y(t)$ is the output signal and $u(t)$ is the input signal. Design, if it is possible, the gain vector $\mathbf{L}$ of a closed loop estimator which puts in -1 as many eigenvalues as possible. The gain vector $\mathbf{L}$ must be designed using the Ackerman formula.

<u>Soluzione.</u> The system is completely observable:

$$\mathcal{O}^- = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 6 & 0 & 4 \end{bmatrix} \qquad \rightarrow \qquad \det \mathcal{O}^- = -2$$

and therefore it is possible to design a closed loop estimator:

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} + \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) - \mathbf{L}\mathbf{y}(t) + \mathbf{B}\mathbf{u}(t)$$

Polynomial $p(s)$ is chosen as follows: $p(s) = (s+1)^3$. The gain vector $\mathbf{L}$ can be determined using the Ackerman formula:

$$\mathbf{L} = -(\mathbf{A} + \mathbf{I})^3 (\mathcal{O}^-)^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

that is

$$\begin{aligned} \mathbf{L} &= -\begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}^3 \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 6 & 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= -\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} -2 & -2 & -0.5 \\ 1 & 0 & 0 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= -\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} -0.5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

The same result can be obtained using the alternative formula. The characteristic polynomial $\Delta_{\mathbf{A}}(s)$ of matrix $\mathbf{A}$, the desired polynomial $p(s)$ and the gain vector $\mathbf{L}$ have the following structure:

$$\Delta_{\mathbf{A}}(s) = s^3 + 4\,s^2 + 6\,s + 4, \qquad p(s) = (s+1)^3 = s^3 + 3\,s^2 + 3\,s + 1,$$

$$\mathbf{L} = \mathbf{P}_c \mathbf{L}_c = \left\{ \begin{bmatrix} 6 & 4 & 1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 6 & 0 & 4 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1 \\ 1 \end{bmatrix}$$

In Matlab:

```
-- Matlab commands -------------------------------------------------------

A=[ -2  1 -1; ...
    -2 -1 -1; ...
     0 -1 -1];
B=[1; 1; -1];
C=[0 1 0];
Omeno=obsv(A,C);                        % Observability matrix
L1=-(A+eye(3))^3*inv(Omeno)*[0;0;1]     % Ackerman formula
L2=-acker(A',C',[-1 -1 -1])'            % Matlab "acker" command (note the "-" sign!)
L3=-place(A',C',[-1 -1.0001 -0.9999])'  % Matlab "place" command (note the "-" sign!)

-- Matlab output ---------------------------------------------------------

 L1 =                  L2 =                  L3 =

    -0.5000               -0.5000               -0.5000
     1.0000                1.0000                1.0000
     1.0000                1.0000                1.0000

-- Matlab commands -------------------------------------------------------

L4=-place(A',C',[-1 -1 -1])'            % The Matlab "place" command has a constraint

-- Matlab output ---------------------------------------------------------

Error using place (line 79)
The "place" command cannot place poles with multiplicity greater than rank(B).

--------------------------------------------------------------------------
```

Example. Let us consider the following discrete linear system:

$$\left\{\begin{array}{rcl}\mathbf{x}(k+1) & = & \begin{bmatrix} -2 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k) \\ y(k) & = & \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\mathbf{x}(k) \end{array}\right. , \qquad \mathbf{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

where $\mathbf{x}(k)$ is the state vector, $y(k)$ is the output signal and $u(k)$ is the input signal.

1.a) Compute the unobservable subspace $\mathcal{E}^-$ of the system. Bring the system in the obserability standard form.

1.b) Compute, if it is possible, the gain vector $\mathbf{L}$ of a closed loop estimator which puts in the origin as many eigenvalues as possible.

<u>Solution.</u> 1.a) The observability matrix of the system is:

$$\mathcal{O}^- = \begin{bmatrix} 1 & 1 & 0 \\ -3 & 1 & 4 \\ 1 & 1 & 0 \end{bmatrix} \qquad \rightarrow \qquad \det \mathcal{O}^- = 0.$$

The determinant of matrix $\mathcal{O}^-$ is zero, and therefore the system is not completely observable. The unobservable subspace $\mathcal{E}^-$ is equal to the kernel of matrix $\mathcal{O}^-$.

$$\mathcal{E}^- = \ker \mathcal{O}^- = \operatorname{span} \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array} \right]$$

```
-- Matlab commands --------------------------------------------------------------

 A=[ -2  1  3; ...
     -1  0  1; ...                                                    Emeno =
     -1  1  2];
 B=[1; 1; 1];                                               -->          -0.5774
 C=[1  1  0];                                                             0.5774
 Omeno=obsv(A,C);     % Observability matrix                             -0.5774
 Emeno=null(Omeno)    % Base for the unobservable subspace

---------------------------------------------------------------------------------
```

The eigenvalues of the unobservable subsystem can be determined bringing the system in the observability standard form using, for example, the following transformation matrix

$$\mathbf{P}^{-1} = \left[\begin{array}{ccc} 1 & 1 & 0 \\ -3 & 1 & 4 \\ \hline 1 & 0 & 0 \end{array} \right], \qquad \mathbf{P} = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -0.25 & 0.25 & 1 \end{array} \right]$$

```
-- Matlab commands --------------------------------------------------------------

 A=[ -2  1  3; ...
     -1  0  1; ...
     -1  1  2];
 B=[1; 1; 1];
 C=[1  1  0];
 Tc=1;                              % Sampling period
 Sys=ss(A,B,C,0,Tc);                % Discrete linear system
 Pmuno=[  1  1  0; ...              % x_new = Pmuno*x_old
         -3  1  4; ...
          1  0  0];
 SysT = ss2ss(Sys,Pmuno)            % State space transformation using matrix Pmuno

-- Matlab output ----------------------------------------------------------------

   a =                                                          b =
               x1          x2          x3                            u1
    x1          0           1    1.66e-16                        x1   2
    x2          1           0   -2.77e-16                        x2   2
    x3       0.25        0.75    1.11e-16                        x3   1

   c =                                                          d =
          x1  x2  x3                                                 u1
    y1     1   0   0                                             y1   0

---------------------------------------------------------------------------------
```

The transformed system has the following form:

$$\begin{cases} \bar{\mathbf{x}}(k+1) &= \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0.25 & 0.75 & 0 \end{array}\right] \mathbf{x}(k) + \left[\begin{array}{c} 2 \\ 2 \\ \hline 1 \end{array}\right] u(k) \\ y(k) &= \left[\begin{array}{cc|c} 1 & 0 & 0 \end{array}\right] \mathbf{x}(k) \end{cases}$$

The eigenvalue of the unobservable subsystem is stable: $\lambda = 0$. So, it is possible to design a closed loop state estimator.

1.b) The gain vector $\mathbf{L}$ can be determined referring to the observability standard form: vector $\bar{\mathbf{L}}$ must be designed such that the eigenvalues of matrix $\mathbf{A}_{11} + \bar{\mathbf{L}}\mathbf{C}_1$ are both located in the origin.

$$\bar{\mathbf{L}} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \qquad \mathbf{A}_{11} + \bar{\mathbf{L}}\mathbf{C}_1 = \begin{bmatrix} l_1 & 1 \\ l_2+1 & 0 \end{bmatrix}$$

The characteristic polynomial of matrix $\mathbf{A}_{11} + \bar{\mathbf{L}}\mathbf{C}_1$ is:

$$\Delta_{\mathbf{A}_{11}+\bar{\mathbf{L}}\mathbf{C}_1}(z) = (z - l_1)z - (l_2 + 1) = z^2 - l_1 z - l_2 - 1$$

Imposing $\Delta_{\mathbf{A}_{11}+\bar{\mathbf{L}}\mathbf{C}_1}(z) = z^2$ one obtains the following solution:

$$\bar{\mathbf{L}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The same result can be obtained using the Ackerman formula:

$$\bar{\mathbf{L}} = -(\mathbf{A}_{11})^2 \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_1\mathbf{A}_{11} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

or using the formula based on the observability canonical form. In this case the characteristic polynomial $\Delta_{\mathbf{A}_{11}}(z)$ of matrix $\mathbf{A}_{11}$ and the desired polynomial $p(z)$ have the following structure:

$$\Delta_{\mathbf{A}_{11}}(z) = z^2 - 1, \qquad p(z) = z^2$$

The obtained vector $\bar{\mathbf{L}}$ is:

$$\bar{\mathbf{L}} = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The final vector $\mathbf{L}$ can be obtained adding a degree of freedom α in the third component of the gain vector and applying the transformation matrix $\mathbf{P}$ which brings the system from the observability canonical form to the original form:

$$\mathbf{L} = \mathbf{P} \begin{bmatrix} \bar{\mathbf{L}} \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -0.25 & 0.25 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ \alpha \end{bmatrix} = \begin{bmatrix} \alpha \\ -\alpha \\ -0.25 + \alpha \end{bmatrix}$$

3) Reduced order state estimator

- The <u>reduced order</u> state estimator is a "simplified" version of the feedback loop state estimator: only the components of the state vector which "are not directly measured" are estimated.

- Let the p rows of matrix $\mathbf{C}$ be linearly independent.

- Let $\mathbf{P}^{-1}$ denote a transformation matrix with the last p rows equal to matrix $\mathbf{C}$:

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{V} \\ \mathbf{C} \end{bmatrix} \qquad \rightarrow \qquad \mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$$

where $\mathbf{V}$ is a free submatrix of dimension $(n-p) \times n$ which must be chosen such that $\mathbf{P}^{-1}$ is a full rank matrix.

- Using $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ one obtains the following transformed system:

$$\overline{\mathbf{x}}(k+1) = \begin{bmatrix} \mathbf{w}(k+1) \\ \mathbf{y}(k+1) \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{A}}_{1,1} & \overline{\mathbf{A}}_{1,2} \\ \overline{\mathbf{A}}_{2,1} & \overline{\mathbf{A}}_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{w}(k) \\ \mathbf{y}(k) \end{bmatrix} + \begin{bmatrix} \overline{\mathbf{B}}_1 \\ \overline{\mathbf{B}}_2 \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = [\ \mathbf{0} \ \ \mathbf{I}_p\ ] \begin{bmatrix} \mathbf{w}(k) \\ \mathbf{y}(k) \end{bmatrix}$$

- The <u>*reduced order state estimator*</u> has the following structure:

$$\boxed{\hat{\mathbf{x}} = \mathbf{P} \begin{bmatrix} \hat{\mathbf{v}}(k) - \mathbf{L}\mathbf{y}(k) \\ \mathbf{y}(k) \end{bmatrix}}$$

were $\hat{\mathbf{v}}(k)$ is the output of the following reduced order dynamic system:

$$\boxed{\begin{aligned} \hat{\mathbf{v}}(k+1) \ = \ & (\overline{\mathbf{A}}_{1,1} + \mathbf{L}\overline{\mathbf{A}}_{2,1})\hat{\mathbf{v}}(k) + \\ & +(\overline{\mathbf{A}}_{1,2} + \mathbf{L}\overline{\mathbf{A}}_{2,2} - \overline{\mathbf{A}}_{1,1}\mathbf{L} - \mathbf{L}\overline{\mathbf{A}}_{2,1}\mathbf{L})\mathbf{y}(k) + \\ & +(\overline{\mathbf{B}}_1 + \mathbf{L}\overline{\mathbf{B}}_2)\mathbf{u}(k) \end{aligned}}$$

- All the eigenvalues of the *reduced order estimator* can be chosen arbitrarily by using the gain vector $\mathbf{L}$ if and only if the couple $(\mathbf{A}, \mathbf{C})$ is observable, that is if the couple $(\overline{\mathbf{A}}_{1,1}, \overline{\mathbf{A}}_{2,1})$ is observable.

Example. Let us consider the following continuous-time linear system

$$\begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -2 \\ -0.5 & -0.5 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ -0.5 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

design the gain vector $\mathbf{L}$ of a "reduced order" state estimator which sets in -1 as many eigenvalues as possible.

Solution. A matrix $\mathbf{P}$ which brings matrix $\mathbf{C}$ to the form $\bar{\mathbf{C}} = \left[\begin{array}{cc|c} 0 & 0 & 1 \end{array}\right]$ is the following:

$$\mathbf{P}^{-1} = \left[\begin{array}{c} \mathbf{V} \\ \hline \mathbf{C} \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \hline -1 & 1 & 0 \end{array}\right] \qquad \rightarrow \qquad \mathbf{P} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using the state space transformation $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ one obtains the following transformed system:

$$\begin{cases} \dot{\bar{\mathbf{x}}}(t) = \left[\begin{array}{cc|c} 0 & -1 & 0.5 \\ -2 & -1 & 1 \\ \hline 0 & -2 & 1 \end{array}\right] \bar{\mathbf{x}}(t) + \left[\begin{array}{c} -0.5 \\ 1 \\ \hline 0 \end{array}\right] u(t) \\ y(t) = \left[\begin{array}{cc|c} 0 & 0 & 1 \end{array}\right] \bar{\mathbf{x}}(t) \end{cases}$$

that is:

$$\begin{cases} \dot{\bar{\mathbf{x}}}(t) = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right] \bar{\mathbf{x}}(t) + \left[\begin{array}{c} \mathbf{B}_1 \\ \hline \mathbf{B}_2 \end{array}\right] u(t) \\ y(t) = \left[\begin{array}{c|c} \mathbf{0} & 1 \end{array}\right] \bar{\mathbf{x}}(t) \end{cases}$$

The vector $\mathbf{L}$ can be computed using the Ackerman formula:

$$\begin{aligned} \mathbf{L} &= -(\mathbf{A}_{11} + \mathbf{I}_2)^2 \begin{bmatrix} \mathbf{A}_{21} \\ \mathbf{A}_{21}\mathbf{A}_{11} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= -\begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}^2 \begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.75 \\ 0.5 \end{bmatrix} \end{aligned}$$

The reduced order observer has the following structure:

$$\hat{\mathbf{x}} = \mathbf{P} \begin{bmatrix} \hat{\mathbf{v}}(t) - \mathbf{L}\, y(t) \\ y(t) \end{bmatrix}$$

where:

$$\dot{\hat{\mathbf{v}}}(t) = [\mathbf{A}_{11} + \mathbf{L}\mathbf{A}_{21}]\hat{\mathbf{v}}(t) + [\mathbf{A}_{12} + \mathbf{L}\mathbf{A}_{22} - \mathbf{A}_{11}\mathbf{L} - \mathbf{L}\mathbf{A}_{21}\mathbf{L}]y(t) + [\mathbf{B}_1 + \mathbf{L}\mathbf{B}_2]u(t)$$

The final dynamic structure is:

$$\dot{\hat{\mathbf{v}}}(t) = \begin{bmatrix} 0 & 0.5 \\ -2 & -2 \end{bmatrix} \hat{\mathbf{v}}(t) + \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} y(t) + \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} u(t)$$