Stability criteria for nonlinear systems

- *First Lyapunov criterion* (reduced method): the stability analysis of an equilibrium point $\mathbf{x}_0$ is done studying the stability of the corresponding linearized system in the vicinity of the equilibrium point.
- *Second Lyapunov criterion* (direct method): the stability analysis of an equilibrium point $\mathbf{x}_0$ is done using proper scalar functions, called *Lyapunov functions*, defined in the state space.

First Lyapunov criterion

- Lyapunov "reduced" criterion. Let us consider the following "continuous time" *["discrete time"]* nonlinear system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\, \mathbf{u}(t)) \qquad\qquad [\ \mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k),\, \mathbf{u}(k))\ ]$$

In the vicinity of the equilibrium point $(\mathbf{x}_0,\, \mathbf{u}_0)$, let us consider the corresponding linearized system:

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\,\tilde{\mathbf{x}}(t) + \mathbf{B}\,\tilde{\mathbf{u}}(t) \qquad\qquad [\ \tilde{\mathbf{x}}(k+1) = \mathbf{A}\,\tilde{\mathbf{x}}(k) + \mathbf{B}\,\tilde{\mathbf{u}}(k)\ ]$$

For the two considered systems, the following statements hold:

1) If all the eigenvalues of matrix $\mathbf{A}$ have "negative real part" *["modulus less than 1"]*, then the equilibrium point $(\mathbf{x}_0,\, \mathbf{u}_0)$ is asymptotically stable also for the nonlinear system.
2) If at least one of the eigenvalues of the matrix $\mathbf{A}$ has "positive real part" *["modulus greater than 1"]*, then the equilibrium point $(\mathbf{x}_0,\, \mathbf{u}_0)$ is unstable also for the nonlinear system.
3) If at least one eigenvalue of the matrix $\mathbf{A}$ is located "on the imaginary axis" *["on the unitary circle"]* while all the other eigenvalues have "negative real part" *["modulus less than 1"]*, then it is not possible to conclude anything about the stability of the equilibrium point $(\mathbf{x}_0,\, \mathbf{u}_0)$ for the nonlinear system. (In this case the criterion is not effective).

Example. Let us consider the following three "autonomous" nonlinear systems:

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 \\ \dot{x}_2 = -x_1^3 - x_2^3 \end{cases} \qquad \begin{cases} \dot{x}_1 = x_1^3 + x_2 \\ \dot{x}_2 = -x_1^3 + x_2^3 \end{cases} \qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 \end{cases}$$

It can be easily verified that the origin $\mathbf{x}_0 = 0$ is an equilibrium point for all the three systems. The Jacobian matrices of the three systems have the following structure:

$$\mathbf{A}_1 = \begin{bmatrix} -3x_1^2 & 1 \\ -3x_1^2 & -3x_2^2 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} 3x_1^2 & 1 \\ -3x_1^2 & 3x_2^2 \end{bmatrix}, \qquad \mathbf{A}_3 = \begin{bmatrix} 0 & 1 \\ -3x_1^2 & 0 \end{bmatrix}$$

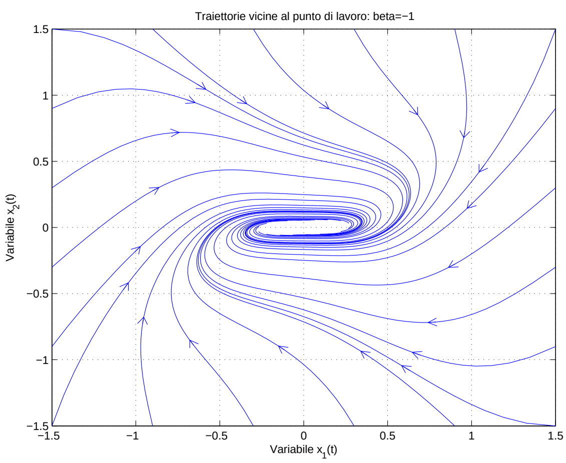
Computing the Jacobian matrices in the origin, one obtains the same matrix $\mathbf{A}$ of the corresponding linearized system:

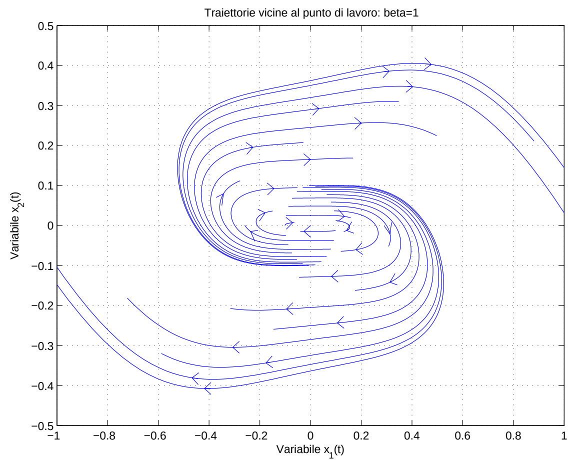
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

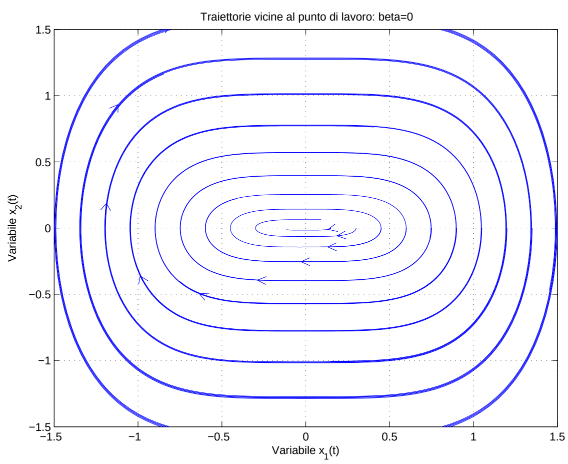
This matrix has two eigenvalues in the origin and therefore the reduced Lyapunov criterion cannot be used. Let us now consider the Lyapunov function $V(\mathbf{x}) = x_1^4 + 2x_2^2$. Computing the derivatives along the trajectories of the nonlinear systems, one obtains:

$$\dot{V} = -4x_1^6 - 4x_2^4 < 0, \qquad \dot{V} = 4x_1^6 + 4x_2^4 > 0, \qquad \dot{V} = 0$$

and applying the "direct" Lyapunov criterion it is possible to prove that in $\mathbf{x}_0 = 0$ the first system is asymptotically stable, the second is unstable, and the third is "simply stable" (see file "exe_pro.m"):







Positive definite functions

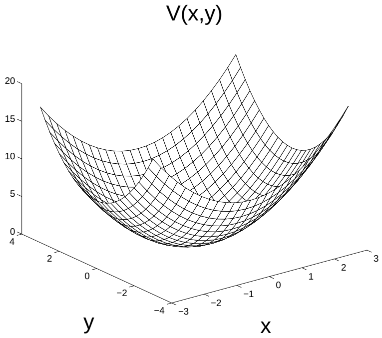
- The "direct" Lyapunov criterion refers to particular "positive definite" or "positive semidefinite" scalar functions, which often have the meaning of "energy functions".

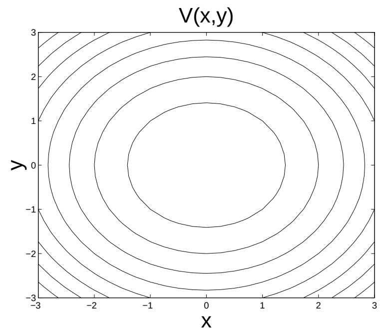
- <u>*Definition.*</u> A continuous function $V(\mathbf{x})$ is "positive definite" (p.d.) *["positive semidefinite" (p.s.d.)]* in the neighborhood of the point $\mathbf{x}_0$, if an open set $W \subseteq \mathbf{R}^n$ of point $\mathbf{x}_0$ exists such that:

  1) $V(\mathbf{x}_0) = 0$;
  2) $V(\mathbf{x}) > 0$ *[*$V(\mathbf{x}) \geq 0$*]* $\mathbf{x} \in W - \mathbf{x}_0$.

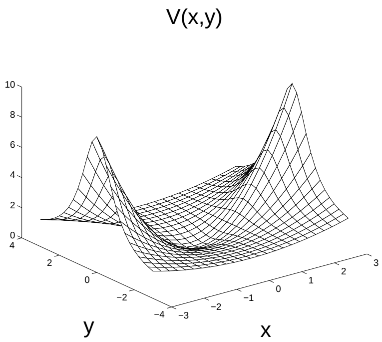
- Examples of "positive definite functions" in point $\mathbf{x}_0 = (0,\ 0)$:

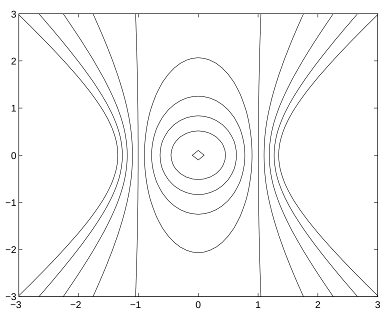
1) $V(x,y) = x^2 + y^2$



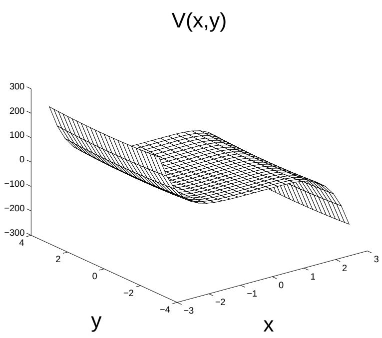


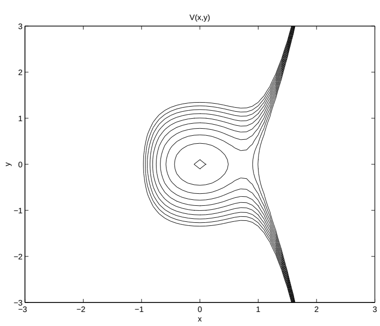
2) $V(x,y) = \dfrac{x^2+y^2}{(1+y^2)}$





3) $V(x,y) = x^2 + y^2 - x^5$





Quadratic forms

- *Definition*. The continuous function $V(\mathbf{x})$ is a *quadratic form* if it can be expressed in the following way:

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{P}\, \mathbf{x}$$

where $\mathbf{P} \in \mathcal{R}^{n\times n}$ is a *symmetric* matrix.

- If the matrix $\mathbf{P}$ is *[semi]*positive definite, then also the corresponding quadratic form is positive *[semi]*definite.

- A symmetric matrix $\mathbf{P}$ *is positive definite if and only if all its principal minors are positive*, that is if the following determinants are positive:

$$p_{1,1} > 0, \quad \begin{vmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{vmatrix} > 0 \; \cdots, \qquad \begin{vmatrix} p_{1,1} & \cdots & p_{1,n} \\ \vdots & & \vdots \\ p_{n,1} & \cdots & p_{n,n} \end{vmatrix} > 0$$

- Let λ_i be the eigenvalues of matrix $\mathbf{P}_s$. *The quadratic form*:

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} = \mathbf{x}^T\mathbf{P}_s\mathbf{x}$$

is positive definite if and only if *all the eigenvalues λ_i are positive. The quadratic form is positive semidefinite* if and only if *all the eigenvalues λ_i are positive or zero:*

$$V(\mathbf{x}) > 0 \;\Leftrightarrow\; \lambda_i > 0 \qquad\qquad V(\mathbf{x}) \geq 0 \;\Leftrightarrow\; \lambda_i \geq 0$$

- Quadratic forms $V(\mathbf{x})$ always refer to symmetric matrices $\mathbf{P}$ because:

  1) a generic matrix $\mathbf{P}$ can always be expressed as the sum of a symmetric matrix $\mathbf{P}_s$ and a skew-symmetric matrix $\mathbf{P}_w$:

$$\mathbf{P} = \frac{\mathbf{P}+\mathbf{P}^T}{2} + \frac{\mathbf{P}-\mathbf{P}^T}{2} = \mathbf{P}_s + \mathbf{P}_w$$

  2) only the symmetric part $\mathbf{P}_s$ has influence on the quadratic $V(\mathbf{x})$:

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} = \mathbf{x}^T\mathbf{P}_s\mathbf{x} + \underbrace{\mathbf{x}^T\mathbf{P}_w\mathbf{x}}_{0} = \mathbf{x}^T\mathbf{P}_s\mathbf{x}$$

  The skew-symmetric matrices $\mathbf{P}_w$ satisfies the following property: the vector $\mathbf{P}_w\mathbf{x}$ is always perpendicular to vector $\mathbf{x}$.

The time derivative of function $V(\mathbf{x})$

- Let us consider the following continuous-time "autonomous" nonlinear system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\, \mathbf{u}_0) \qquad \rightarrow \qquad \dot{\mathbf{x}}(t) = \bar{\mathbf{f}}(\mathbf{x}(t))$$

and let W be a neighborhood of the equilibrium point $\mathbf{x}_0$ corresponding to the constant input $\mathbf{u}(t) = \mathbf{u}_0$. Let $V(\mathbf{x})$ be a continuous scalar function with continuous first derivatives defined in the neighborhood W:

$$V(\mathbf{x}) : W \rightarrow \mathcal{R}$$

- The gradient of function $V(\mathbf{x})$ is a vector defined as follows:

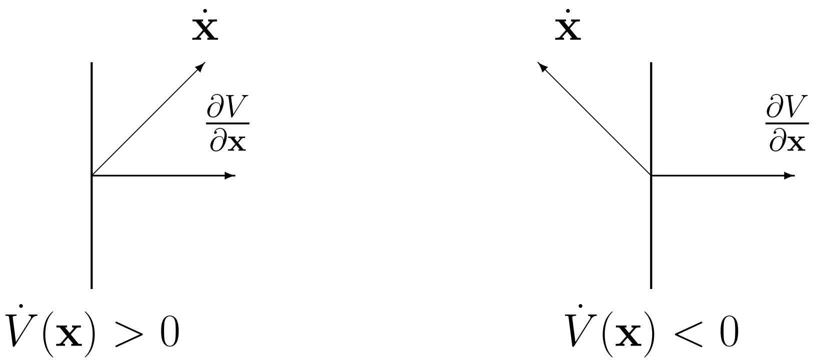
$$\frac{\partial V}{\partial \mathbf{x}} \triangleq \left[\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right] = \text{grad } (V)$$

This vector has the geometric meaning of *"direction in $\mathbf{R}^n$ along which the function $V(\mathbf{x})$ increases with the highest slope"*.

- If $\mathbf{x}(t)$ is a solution of the nonlinear system $\dot{\mathbf{x}}(t) = \bar{\mathbf{f}}(\mathbf{x}(t))$, then the time-derivative $\dot{V}(\mathbf{x})$ of function $V(\mathbf{x})$ can be expressed as follows:

$$\dot{V}(\mathbf{x}) = \frac{\partial V}{\partial \mathbf{x}}\dot{\mathbf{x}}(t) = \frac{\partial V}{\partial \mathbf{x}}\bar{\mathbf{f}}(\mathbf{x}(t)) = \frac{\partial V}{\partial x_1}\bar{f}_1(\mathbf{x}) + \cdots + \frac{\partial V}{\partial x_n}\bar{f}_n(\mathbf{x})$$

The function $\dot{V}(\mathbf{x})$ is the scalar product of the two vectors $\frac{\partial V}{\partial \mathbf{x}}$ and $\dot{\mathbf{x}}$, and therefore it can be interpreted in a geometric way:



- *Note*: the computation of $\dot{V}(\mathbf{x})$ does not require the knowledge of the trajectory $\mathbf{x}(t)$, that is it does not require the explicit solution $\mathbf{x}(t)$ of the nonlinear differential equations of the system.

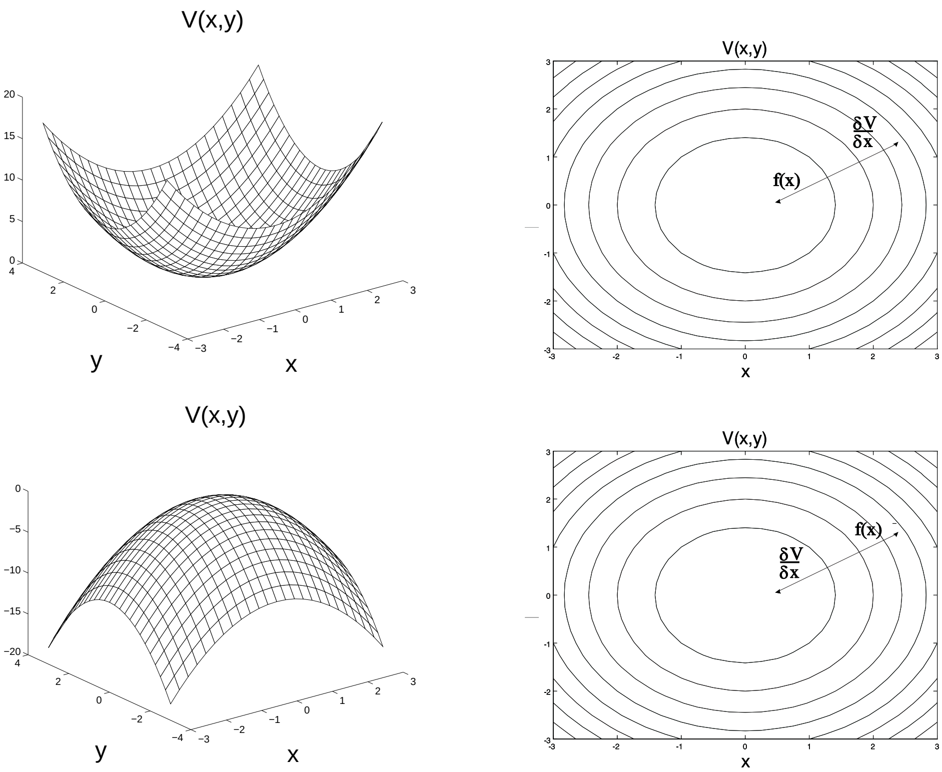
Second Lyapunov criterion

- *"Direct" Lyapunov criterion*. Let us consider the following continuous-time nonlinear system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\, \mathbf{u}_0) \qquad \rightarrow \qquad \dot{\mathbf{x}}(t) = \overline{\mathbf{f}}(\mathbf{x}(t))$$

  and let $\mathbf{x}_0$ be an equilibrium point corresponding to the constant input $\mathbf{u}_0$.
  1) If in a neighborhood W of $\mathbf{x}_0$ exists a function $V(\mathbf{x}) : W \rightarrow \mathcal{R}$ *positive definite* with continuous first derivatives and if $\dot{V}(\mathbf{x})$ is *negative semidefinite*, then the point $\mathbf{x}_0$ is *stable* for the nonlinear system.
  2) If $\dot{V}(\mathbf{x})$ is *negative definite*, then the point $\mathbf{x}_0$ is *asymptotically* stable.

- The stability of point $\mathbf{x}_0$ is guaranteed only if both the conditions $V(\mathbf{x}) > 0$ and $\dot{V}(\mathbf{x}) < 0$ are satisfied:



- ***La Salle-Krasowskii stability criterion***. Let us consider the following continuous-time nonlinear system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\, \mathbf{u}_0) \qquad \rightarrow \qquad \dot{\mathbf{x}}(t) = \overline{\mathbf{f}}(\mathbf{x}(t))$$

  and let $\mathbf{x}_0$ be an equilibrium point corresponding to the constant input $\mathbf{u}_0$. If:

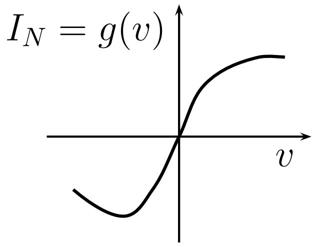
  1) in a neighborhood W of $\mathbf{x}_0$ it exists a function $V(\mathbf{x}) : W \rightarrow \mathcal{R}$ *positive definite* with continuous first derivatives;
  2) $\dot{V}(\mathbf{x})$ is *negative semidefinite*;
  3) the set $\mathcal{N} = \{\mathbf{x} \in W | \dot{V} = 0\}$ *does not contain perturbed trajectories*;

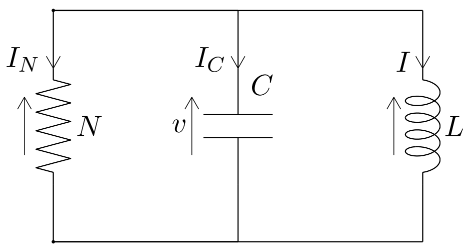
  then $\mathbf{x}_0$ is an equilibrium point *asymptotically* stable for the given nonlinear system.

- The La Salle-Krasowskii stability criterion is a "refinement" of the direct Lyapunov criterion. Often, it allows to prove the asymptotic stability of an equilibrium point $\mathbf{x}_0$ also when the Lyapunov criterion guarantees only the simple stability.

Example. Let us consider the electric circuit shown below where N is a nonlinear element with a current-voltage characteristics $I_N = g(v)$, such that function $g(v)$ satisfies the relation: $vg(v) > 0$. Let us consider the state vector: $\mathbf{x} = [I,\ v]^T$. The system is described by the following differential equations:

$$\begin{cases} \dot{I} = \dfrac{v}{L} \\ \dot{v} = -\dfrac{g(v)}{C} - \dfrac{I}{C} \end{cases}$$





The origin is an equilibrium point for the system. The stability of the equilibrium point can be studied using the following positive definite function $V(\mathbf{x})$:

$$V(\mathbf{x}) = \frac{1}{2}LI^2 + \frac{1}{2}Cv^2$$

which represents the energy stored in the capacitor and the inductance of the electrical circuit. Since the function:

$$\dot{V}(\mathbf{x}) = LI \underbrace{\frac{v}{L}}_{\dot{I}} + Cv \underbrace{\left(-\frac{g(v)}{C} - \frac{I}{C}\right)}_{\dot{v}} = -v\, g(v) \le 0$$

is negative semidefinite, the equilibrium point $\mathbf{x} = 0$ is surely *stable*. To prove the asymptotic stability of the equilibrium point $\mathbf{x} = 0$ one can try to use the La Salle-Krasowskii criterion. The set $\mathcal{N}$ of the points where function $\dot{V}(\mathbf{x})$ is zero is the following:

$$\mathcal{N} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad\qquad I \in \mathcal{R}$$

A system trajectory is completely contained within $\mathcal{N}$ if and only if $v(t) = 0$. Using this condition in the second differential equation of the system, one obtains:

$$0 = -\frac{I}{C} \qquad \rightarrow \qquad I = 0$$

The only trajectory contained within $\mathcal{N}$ which satisfies the differential equations of the system is $\mathbf{x} = 0$. Therefore $\mathcal{N}$ does not contain perturbed trajectories of the considered nonlinear system. Applying the La Salle-Krasowskii criterion, one can conclude that the origin is asymptotically stable.

The same result could also be obtained using the reduced Lyapunov criterion. In this case the system Jacobian is the following:

$$\mathbf{A}(I, v) = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{C}\frac{\partial g(v)}{\partial v} \end{bmatrix} \qquad \rightarrow \qquad \mathbf{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{\alpha}{C} \end{bmatrix}$$

where $\alpha = \left.\frac{\partial g(v)}{\partial v}\right|_{v=0} > 0$ denotes the slope of function $g(v)$ in the point $v = 0$. The characteristic polynomial of matrix $\mathbf{A}$ is the following:

$$\det(s\mathbf{I} - \mathbf{A}) = s\left(s + \frac{\alpha}{C}\right) + \frac{1}{LC} = s^2 + \frac{\alpha}{C}s + \frac{1}{LC}$$

Since $\alpha > 0$, $C > 0$ and $L > 0$, the eigenvalues of matrix $\mathbf{A}$ have surely a negative real part and therefore the point $\mathbf{x} = 0$ is an asymptotically stable equilibrium point for the considered nonlinear system.

- *Instability Lyapunov criterion*. Let us consider the following continuous-time nonlinear system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\, \mathbf{u}_0) \qquad \rightarrow \qquad \dot{\mathbf{x}}(t) = \overline{\mathbf{f}}(\mathbf{x}(t))$$

and let $\mathbf{x}_0$ be an equilibrium point corresponding to the constant input $\mathbf{u}_0$. If:

1) in a neighborhood W of $\mathbf{x}_0$ it exists a continuous function $V(\mathbf{x}) : W \rightarrow \mathcal{R}$ with continuous first derivatives such that $V(\mathbf{x}_0) = 0$;
2) the point $\mathbf{x}_0$ is an accumulation point for the set of the point $\mathbf{x} \in W$ for which it is $V(\mathbf{x}) > 0$;
3) $\dot{V}(\mathbf{x})$ is *positive definite* in W;

then $\mathbf{x}_0$ is an *unstable* equilibrium point.

- *Note*: this criterion can be used also if function $V(\mathbf{x})$ is NOT positive definite in W.

Example. For the following system

$$\begin{cases} \dot{x}_1 &= x_1 - x_1 x_2 \\ \dot{x}_2 &= -x_2 + x_1 x_2 \end{cases}$$

$\mathbf{x} = 0$ is equilibrium point. Let us consider the following function

$$V(\mathbf{x}) = x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2)$$

which is positive within the region

$$W^+ = \{(x_1,\, x_2) \mid x_1 > x_2,\, x_1 > -x_2\}$$

The origin is clearly an accumulation point for the set W^+. The function

$$\dot{V}(\mathbf{x}) = 2x_1\dot{x}_1 - 2x_2\dot{x}_2 = 2x_1^2(1 - x_2) + 2x_2^2(1 - x_1) > 0$$

is positive definite in W^+. Applying the instability Lyapunov criterion that the equilibrium point $\mathbf{x} = 0$ is unstable.

Stability of the discrete nonlinear systems

- Let us consider the following discrete-time nonlinear system:

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k),\, \mathbf{u}_0) \qquad \to \qquad \mathbf{x}(k+1) = \overline{\mathbf{f}}(\mathbf{x}(k))$$

and let $\mathbf{x}_0$ be an equilibrium point corresponding to the constant input $\mathbf{u}_0$.

- To apply the "direct" Lyapunov criterion to a "discrete" nonlinear system one has to refer to a <u>continuous</u> function $V(\mathbf{x}) : W \to \mathcal{R}$ defined in a neighborhood W of the point $\mathbf{x}_0$, but in this case it is not possible to compute the function $\dot{V}(\mathbf{x})$ along the system trajectories because in this case the trajectories $\mathbf{x}(k)$ are discrete.

- In this case the following discrete function must be used:

$$\Delta V(\mathbf{x}(k)) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) = V(\overline{\mathbf{f}}(\mathbf{x}(k))) - V(\mathbf{x}(k))$$

which represents the one step increment of the function $V(\mathbf{x})$ computed along the trajectories $\mathbf{x}(k)$ of the system.

Example. Let us consider the following discrete nonlinear system which has $\mathbf{x}_0 = 0$ as equilibrium point, and let $V(\mathbf{x})$ be a proper positive definite function defined in a neighborhood of the origin:

$$\begin{cases} x_1(k+1) &= \dfrac{x_2(k)}{1+x_2^2(k)} \\[2mm] x_2(k+1) &= \dfrac{x_1(k)}{1+x_2^2(k)} \end{cases} \qquad \begin{cases} V(\mathbf{x}) = x_1^2 + x_2^2 \\[2mm] \Delta V(\mathbf{x}) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \end{cases}$$

The function $\Delta V(\mathbf{x})$ can be computed as follows:

$$\Delta V(\mathbf{x}) = \left(\frac{x_2}{(1+x_2^2)}\right)^2 + \left(\frac{x_1}{(1+x_2^2)}\right)^2 - x_1^2 - x_2^2 = \frac{-(2+x_2^2)x_2^2}{(1+x_2^2)^2}(x_1^2+x_2^2) \le 0$$

The function $\Delta V(\mathbf{x})$ is negative semidefinite, and therefore the system is stable. The set N of all the point where $\Delta V(\mathbf{x}) = 0$ is $N = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$. Substituting $x_2 = 0$ within the second difference equation one obtains $x_1 = 0$, that is $\mathbf{x} = 0$ is the only trajectory of the system contained within the set N. Using the La Salle-Krasowskii criterion it is possible to state that the equilibrium point $\mathbf{x} = 0$ is asymptotically stable for the given nonlinear system.

Stability and instabili criteria for the discrete systems

- Let us consider the following discrete-time nonlinear system:

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k),\, \mathbf{u}_0) \qquad \rightarrow \qquad \mathbf{x}(k+1) = \overline{\mathbf{f}}(\mathbf{x}(k))$$

  an let $\mathbf{x}_0$ be an equilibrium point corresponding to the constant input $\mathbf{u}_0$. For a nonlinear discrete system the following three criteria hold.

- *Property.* ["Direct" Lyapunov criterion] If in a neighborhood W of point $\mathbf{x}_0$ it exists a continuous *positive definite* function $V(\mathbf{x}) : W \rightarrow \mathcal{R}$ and if the function $\Delta V(\mathbf{x})$ is *negative semidefinite*, then the point $\mathbf{x}_0$ is *stable*. If the function $\Delta V(\mathbf{x})$ is *negative definite*, then the point $\mathbf{x}_0$ is *asymptotically stable*.

- *Property.* [La Salle-Krasowskii stability criterion.] If in a neighborhood W of the point $\mathbf{x}_0$ it exists a continuous *positive definite* function $V(\mathbf{x}) : W \rightarrow \mathcal{R}$, if the function $\Delta V(\mathbf{x})$ is *negative semidefinite* and if the set $\mathcal{N} = \{\mathbf{x} \in W | \Delta V(\mathbf{x}) = 0\}$ does not contain perturbed trajectories of the given system, then $\mathbf{x}_0$ is an equilibrium point *asymptotically stable*.

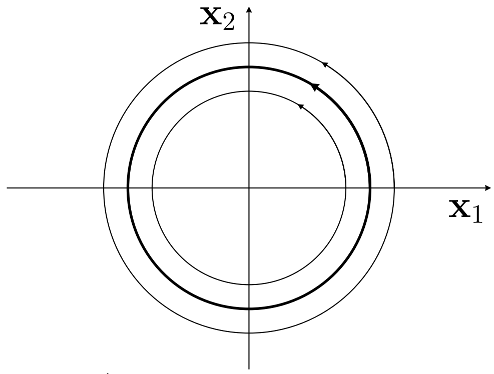
- *Property.* [Instability Lyapunov criterion.] If in a neighborhood W of the point $\mathbf{x}_0$ it exists a continuous function $V(\mathbf{x}) : W \rightarrow \mathcal{R}$ which is zero in $\mathbf{x}_0$, if $\mathbf{x}_0$ is an accumulation point for the set of all the points $\mathbf{x}$ for which $V(\mathbf{x}) > 0$, and if $\Delta V(\mathbf{x})$ is positive definite in W, then $\mathbf{x}_0$ is an *unstable* equilibrium point for the given nonlinear system.

Example. Let us consider the following autonomous system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The free evolution starting from the initial condition $\mathbf{x}_0$ is the following

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$



The trajectory in the state space is a circle with radius $r = 1$.
A linear system can have periodic free evolutions and these trajectories are always stable. The linear system cannot have periodic closed trajectories which are asymptotically stable. This situation can happen only for nonlinear systems.

Let us consider, for example, the following non linear system:

$$\begin{cases} \dot{x}_1 &= -x_2 + x_1(1-r^2) \\ \dot{x}_2 &= x_1 + x_2(1-r^2) \end{cases} \qquad \text{dove} \qquad r^2 = x_1^2 + x_2^2$$

One can easily verify by substitution that $r = 1$ is a periodic solution (a "limit cycle") of the give nonlinear system. The "stability" o "instability" of the limit cycle $r = 1$ can be determined using a state space transformation. Using the polar coordinates

$$\begin{cases} x_1 &= r\cos\theta \\ x_2 &= r\sin\theta \end{cases}$$

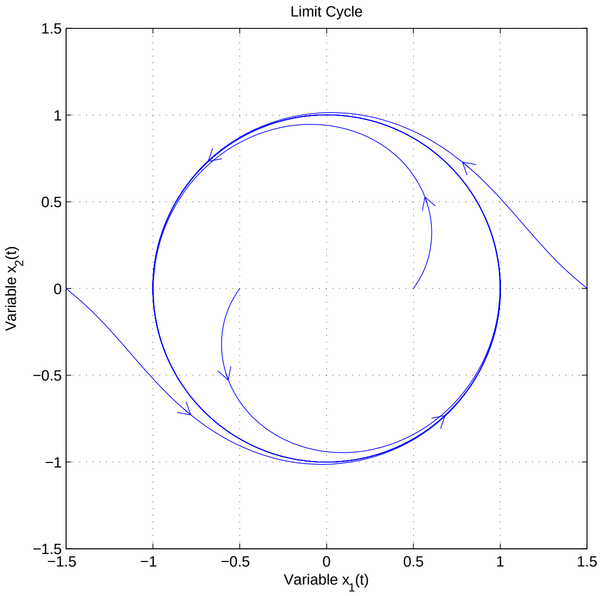
the given system transforms as follows

$$\begin{cases} \dot{r}\cos\theta - r\dot{\theta}\sin\theta &= -r\sin\theta + r\cos\theta(1-r^2) \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta &= r\cos\theta + r\sin\theta(1-r^2) \end{cases}$$

Combining the two equations one obtains the following equivalent dynamic system with separated variables:

$$\begin{cases} \dot{r} &= r(1-r^2) \\ \dot{\theta} &= 1 \end{cases} \qquad \rightarrow \qquad \begin{cases} r(t) &= \frac{e^t r_0}{\sqrt{1+(e^{2t}-1)r_0^2}} \\ \theta(t) &= t + \theta_0 \end{cases}$$

The qualitative behavior of the system trajectories in the plane $(x_1,\ x_2)$ is the following (see files: "ciclo_limite.m" and "ciclo_limite_ode.m"):



So, the limit cycle $r = 1$ is asymptotically stable, that is all the system trajectories (except the trajectory $r = 0$) asymptotically tend to the limit cycle $r = 1$, regardless of the initial condition. An alternative way to prove that the limit cycle $r = 1$ is asymptotically stable is to consider the following function $V(r)$ which is positive definite in the vicinity of point $r = 1$:

$$V(r) = \frac{1}{2}(1 - r)^2$$

Its time derivative is a function which is negative definite in the vicinity of the point $r = 1$:

$$\dot{V}(r) = -(1 - r)\dot{r} = -r(1 - r)^2(1 + r) < 0$$

So it follows that the limit cycle $r = 1$ is asymptotically stable.

Example. Let us consider the following continuous-time nonlinear system:

$$\begin{cases} \dot{x}_1 = \beta(2 - x_1) + x_1^2 x_2 \\ \dot{x}_2 = x_1 - x_1^2 x_2 \end{cases}$$

1.a) Compute, for variable $\beta > 0$, the equilibrium points of the system;

1.b) Study, for variable $\beta > 0$, the stability of the equilibrium points.

<u>Solution</u>. 1.a) The equilibrium points can be determined solving the following system:

$$\begin{cases} 0 = \beta(2 - x_1) + x_1^2 x_2 \\ 0 = x_1 - x_1^2 x_2 = x_1(1 - x_1 x_2) \end{cases}$$

The second equation is solved by

$$x_1 = 0 \qquad \text{and for} \qquad x_1 x_2 = 1$$

The solution $x_1 = 0$ does not satisfy the first equation. Substituting $x_1 x_2 = 1$ in the first equation one obtains:

$$2\beta - \beta x_1 + x_1 = 0 \qquad \rightarrow \qquad x_1 = \frac{2\beta}{\beta - 1}$$

For $\beta \neq 1$, the only equilibrium point of the system is:

$$\bar{x}_1 = \frac{2\beta}{\beta - 1} \qquad \bar{x}_2 = \frac{\beta - 1}{2\beta}$$

1.b) Linearizing in the vicinity of this point one obtains:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\beta + 2x_1x_2 & x_1^2 \\ 1 - 2x_1x_2 & -x_1^2 \end{bmatrix}_{(\bar{x}_1, \bar{x}_2)} \mathbf{x}(t) = \begin{bmatrix} 2 - \beta & \dfrac{4\beta^2}{(\beta-1)^2} \\ -1 & \dfrac{-4\beta^2}{(\beta-1)^2} \end{bmatrix} \mathbf{x}(t)$$

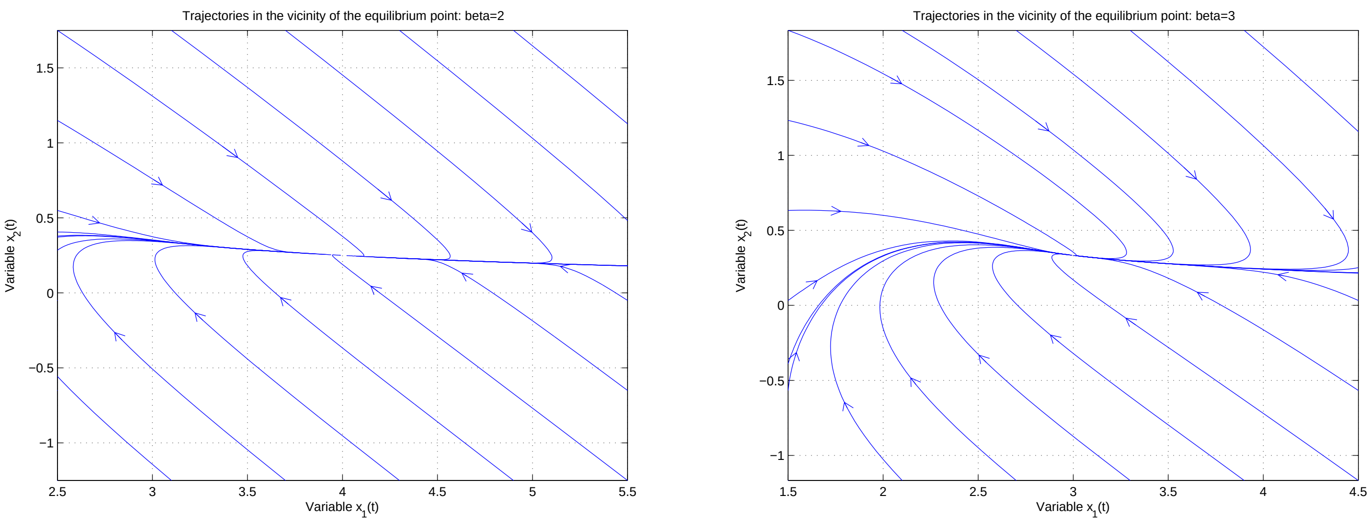
The characteristic polynomial of the system is:

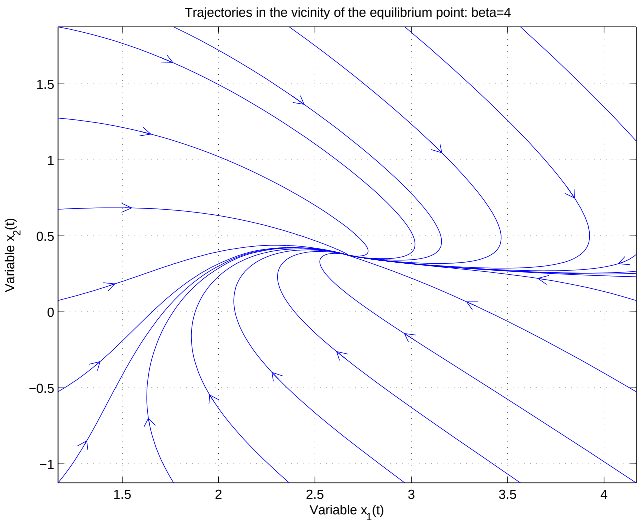
$$\Delta(s) = s^2 + \left[\beta - 2 + \frac{4\beta^2}{(\beta-1)^2}\right] s + \frac{4\beta^2}{(\beta-1)} = 0$$

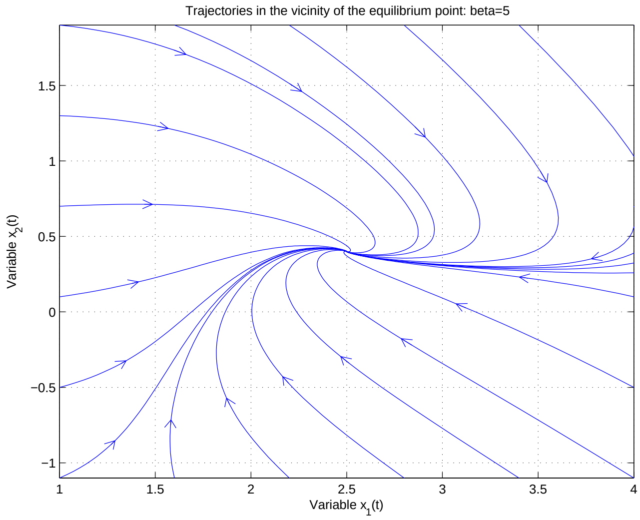
from which one obtains

$$\Delta(s) = s^2 + \frac{\beta^3 + 5\beta - 2}{(\beta-1)^2} s + \frac{4\beta^2}{(\beta-1)} = 0$$

The equilibrium point is stable if the coefficients of this polynomial are both positive. This happens for $\beta > 1$. For $\beta < 1$, at least one eigenvalue of the system is unstable and therefore the equilibrium point is unstable. The system trajectories for $\beta = 2$, $\beta = 3$, $\beta = 4$ and $\beta = 5$ are shown in the following figures:







These simulations have been obtained in the Matlab environment using the following command file "exe_x1x2.m":

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function exe_x1x2
% Nonlinear system:
% x1d=beta*(2-x1)+x1^2*x2
% x2d=x1-x1^2*x2
%%%%%%%%%%%%%%%%
MainString = mfilename; Stampa=0;
global beta
ii=0;
for beta=(2:5);                     % Changing beta changes the equilibrium point
   x10=2*beta/(beta-1);             % Equilibrium point
   x20=(beta-1)/(2*beta);
   ii=ii+1;
   figure(ii); clf
   V=[[-1.5 1.5]+x10 [-1.5 1.5]+x20];   % Plot window
   In_Con=inicond(V,[5,5]);             % Initial conditions
   Tspan=(0:0.005:1)*2;                 % Simulation final time
   fr=10;  dx=0.06; dy=dx;              % Arrow position and arrow width
   for jj=(1:size(In_Con,1))
      [t,x]=ode23(@exe_x1x2_ode,Tspan,In_Con(jj,:));       % ODE simulation
      plot(x(:,1),x(:,2)); hold on                          % Plot
      freccia(x(fr,1),x(fr,2),x(fr+1,1),x(fr+1,2),dx,dy)    % Draw the arrows
   end
   grid on;  axis(V)                        % Grid and axis
   xlabel('Variable x_1(t)')               % Label along axis x
   ylabel('Variable x_2(t)')               % Label along axis y
   title(['Trajectories in the vicinity of the equilibrium point: beta=' num2str(beta)])
  if Stampa;  eval(['print -depsc ' MainString '_' num2str(gcf) '.eps']);  end
end
return

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 function dx=exe_x1x2_ode(t,x);
 global beta
 dx(1,1)=beta*(2-x(1))+x(1)^2*x(2);
 dx(2,1)=x(1)-x(1)^2*x(2);
 return
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Example. Let us consider the following continuous-time nonlinear system:

$$\begin{cases} \dot{x}_1 &= x_1 - x_1 x_2 \\ \dot{x}_2 &= x_1 x_2 - x_2 \end{cases}$$

1.a) Compute the equilibrium points of the system and study the stability of these point using the reduced Lyapunov criterion;

1.b) If necessary, to complete the stability analysis of point 1.a, use the following function: $V(x_1, x_2) = x_1 + x_2 - \ln x_1 - \ln x_2 - 2$.

Solution. 1.a) The equilibrium points can be determined imposing $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$

$$\begin{cases} 0 &= x_1 - x_1 x_2 \\ 0 &= x_1 x_2 - x_2 \end{cases} \qquad \rightarrow \qquad \begin{cases} x_1(1 - x_2) &= 0 \\ x_2(x_1 - 1) &= 0 \end{cases}$$

The two possible equilibrium points are

$$(x_1,\ x_2) = (0,\ 0), \qquad\qquad (x_1,\ x_2) = (1,\ 1)$$

The Jaconian of the system in the point $(0,\ 0)$ is

$$J_0 = \begin{bmatrix} 1 - x_2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix}_{(x_1,\, x_2)=(0,\, 0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since the Jacobian J_0 has an unstable eigenvalue $\lambda = 1$, the equilibrium point $(x_1,\ x_2) = (0,\ 0)$ is surely unstable. The value of the system Jacobian in point $(1,\ 1)$ is

$$J_1 = \begin{bmatrix} 1 - x_2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix}_{(x_1,\, x_2)=(1,\, 1)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Since the Jacobian J_1 is characterized by two imaginary eigenvalues, the reduced Lyapunov criterion cannot be used to study the stability of the equilibrium point $(x_1,\ x_2) = (1,\ 1)$.

1.b) To complete the stability analysis of the equilibrium point $(1,\ 1)$ let us use the given function

$$V(x_1, x_2) = x_1 + x_2 - \ln x_1 - \ln x_2 - 2$$

In the vicinity of the point $(1,\ 1)$, this function is positive definite. In fact

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

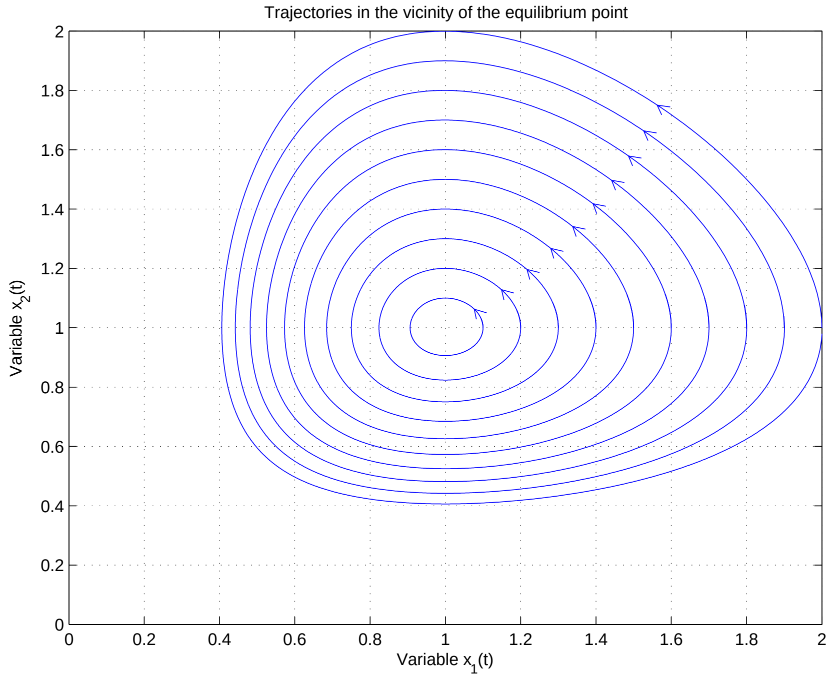
Its time derivative is

$$\dot{V}(x_1, x_2) = \text{grad}V\ \dot{\mathbf{x}} = \left(1 - \frac{1}{x_1}\right)\dot{x}_1 + \left(1 - \frac{1}{x_2}\right)\dot{x}_2$$

from which, by substitution, one obtains

$$\dot{V}(x_1, x_2) = x_1 - x_1x_2 + x_1x_2 - x_2 - (1 - x_2) - (x_1 - 1) = 0$$

It follows that $(1,\ 1)$ is an equilibrium point simply stable: the trajectories move along the level curves of the function $V(x_1, x_2)$. The system trajectories in the plane $(x_1,\ x_2)$ are the following (see file "exe_lnx.m"):



Example. Let us consider the following nonlinear differential equation:

$$\ddot{y}(t) = \cos y(t) - \frac{3}{2\pi}\, y(t) - \beta\dot{y}(t)$$

1.a) Write the state space equations of the considered dynamic system and find the equilibrium point of the system;

1.b) Compute the values of parameter β for which the nonlinear system is asymptotically stable in the vicinity of the equilibrium point;

Solution. 1.a) Setting $x_1 = y$ and $x_2 = \dot{y}$, the state space equations of the given nonlinear system are the following:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \cos x_1 - \frac{3}{2\pi}\, x_1 - \beta x_2 \end{cases}$$

The equilibrium points can be determined imposing $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$:

$$x_2 = 0, \qquad\qquad \cos x_1 = \frac{3}{2\pi}\, x_1$$

From these relations one obtains the following equilibrium point:

$$x_1 = \frac{\pi}{3}, \qquad\qquad x_2 = 0.$$

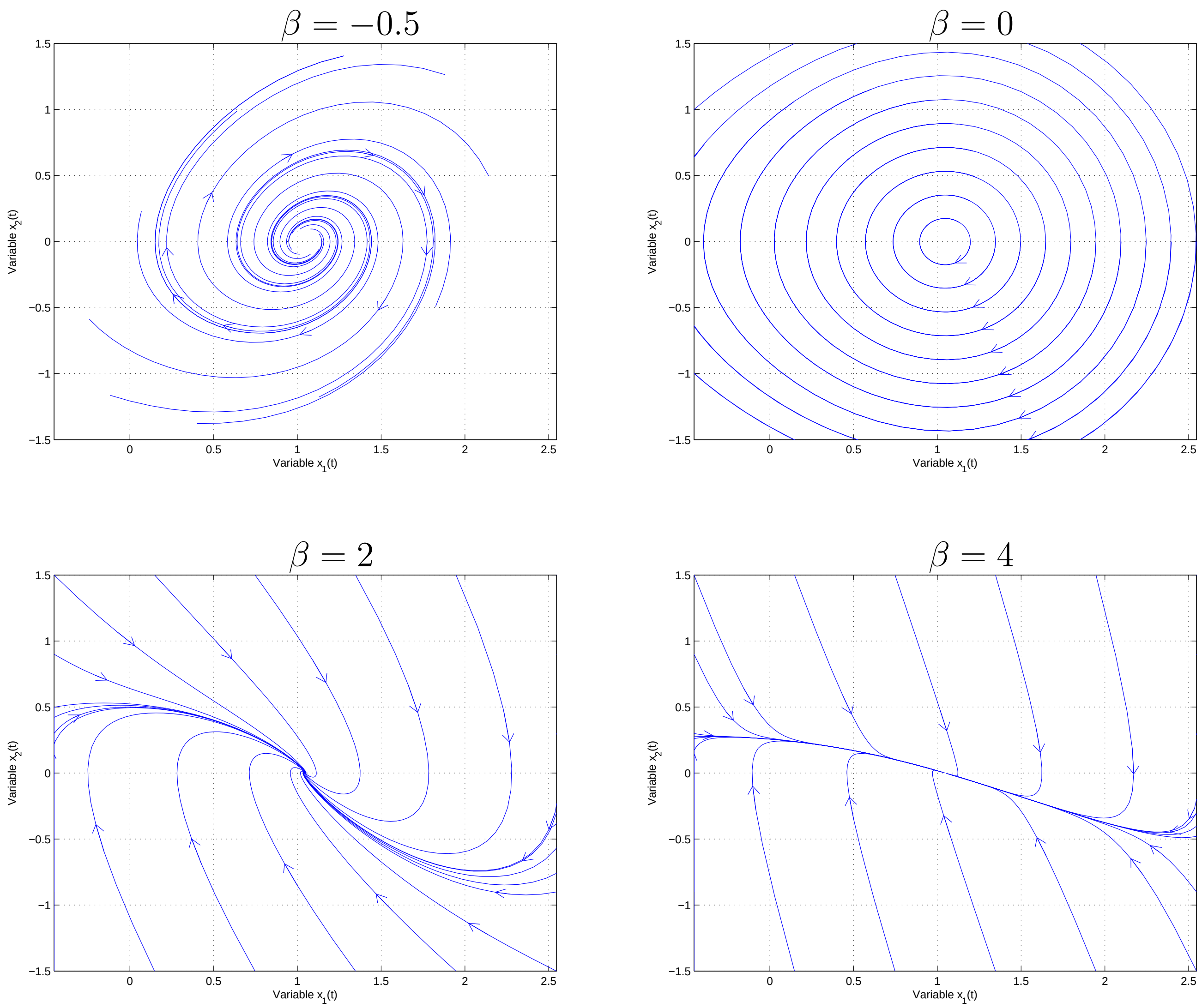
1.b) Linearizing in the vicinity of the equilibrium point one obtains:

$$\mathbf{J} = \left[\begin{array}{cc} 0 & 1 \\ -\sin x_1 - \frac{3}{2\pi} & -\beta \end{array}\right]_{x_1=\frac{\pi}{3}} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{\sqrt{3}}{2} - \frac{3}{2\pi} & -\beta \end{array}\right]$$

The characteristic polynomial of this matrix is:

$$s^2 + \beta\, s + \frac{\sqrt{3}}{2} + \frac{3}{2\pi} = 0.$$

The system is asymptotically stable in the vicinity of the equilibrium point if $\beta > 0$, while it is unstable if $\beta < 0$. For $\beta = 0$ the reduced Lyapunov criterion cannot be used because the matrix $\mathbf{J}$ has two complex eigenvalues with zero real part. The system trajectories for $\beta = -0.5$, $\beta = 0$, $\beta = 2$ and $\beta = 4$ are shown in the following figures (see file "exe_cosx.m"):



Example. Let us consider the following discrete nonlinear system:

$$\begin{cases} x_1(k+1) &= \alpha x_1(k) - x_1^3(k) \\ x_2(k+1) &= -x_2^3(k) - x_1^3(k) \end{cases}$$

1.a) Study the stability of the equilibrium point $\mathbf{x} = 0$, for variable α, using the reduced Lyapunov criterion;

1.b) Study the stability of the equilibrium point $\mathbf{x} = 0$ using the function $V(x_1, x_2) = x_1^2 + x_2^2$.

Solution. 1.a) Linearizing in the vicinity of the equilibrium point $\mathbf{x} = 0$ one obtains:

$$\mathbf{J} = \begin{bmatrix} \alpha - 3x_1^2 & 0 \\ -3x_1^2 & -3x_2^2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$$

The characteristic polynomial of matrix $\mathbf{J}$ is:

$$z(z-\alpha) = 0 \qquad \to \qquad z_1 = 0, \;\; z_2 = \alpha.$$

The system is asymptotically stable in $\mathbf{x} = 0$ if $|\alpha| < 1$, while it is unstable if $|\alpha| > 1$. For $|\alpha| = 1$ the reduced Lyapunov criterion cannot be used because one of the two eigenvalues of matrix $\mathbf{J}$ is located on the unitary circle.

1.b) The function:

$$V(x_1, x_2) = x_1^2 + x_2^2$$

is positive definite in a neighborhood of $\mathbf{x} = 0$ and therefore $V(x_1, x_2)$ is a possible Lyapunov function. The computation of $\Delta V(x_1, x_2)$ when $\alpha = 1$ is the following:

$$\begin{aligned} \Delta V(\mathbf{x}) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\ &= x_1^2(1-x_1^2)^2 + (-x_2^3 - x_1^3)^2 - x_1^2 - x_2^2 \\ &= x_1^2 - 2x_1^4 + x_1^6 + x_2^6 + 2x_1^3x_2^3 + x_1^6 - x_1^2 - x_2^2 \\ &= -x_1^4[2 - 2x_1^2] - x_2^2[1 - x_2^4 - 2x_2x_1^3] < 0 \end{aligned}$$

The obtained function is negative definite in the vicinity of $\mathbf{x} = 0$ and therefore, when $\alpha = 1$, the given nonlinear system is asymptotically stable in $\mathbf{x} = 0$.
When $\alpha = -1$, the given function cannot be used to study the stability of the point $\mathbf{x} = 0$. In this case the first equation of the system becomes

$$x_1(k+1) = -x_1(k) - x_1^3(k)$$

This equation does not depend on the variable x_2. Let us now consider the function $V(x_1) = x_1^2$ and compute $\Delta V(x_1)$:

$$\begin{aligned} \Delta V(x_1) &= (-x_1 - x_1^3)^2 - x_1^2 \\ &= x_1^2 + 2x_1^4 + x_1^6 - x_1^2 = 2x_1^4 + x_1^6 > 0 \end{aligned}$$

Function $\Delta V(x_1)$ is positive definite in $\mathbf{x} = 0$ and therefore, for the instability Lyapunov criterion, one can conclude that for $\alpha = -1$ the nonlinear system is unstable in $\mathbf{x} = 0$.

Example. Given the following discrete nonlinear system:

$$y(k+1) = -(r-2)y(k) - r\,y^2(k)$$

1.a) Find, for $r > 0$, the equilibrium points of the system and study their stability using the reduced Lyapunov criterion.

1.b) For $r = 3$, prove that the equilibrium point $\mathbf{x} = 0$ is asymptotically stable using the Lyapunov function $V(y) = y^2 + \alpha y^3$ and properly choosing the value of parameter α. For $r = 1$, prove that $\mathbf{x} = 0$ is unstable using the function $V(y) = -y$.

Solution. 1.a) The equilibrium point of the system can be obtained imposing $y(k+1) = y(k)$:

$$y(k) = -y(k)[r - 2 + r\,y(k)].$$

The equilibrium points of the system are:

$$y_1 = 0, \qquad\qquad y_2 = \frac{1-r}{r}.$$

The linearized system in the vicinity of the point $y_1 = 0$ is:

$$y(k+1) = -[r - 2 + 2ry(k)]_{(y=0)}\, y(k) = (2-r)y(k)$$

The equilibrium point $y_1 = 0$ is stable for $|2 - r| < 1$, that is for $1 < r < 3$, and it is unstable for $r > 3$ and $r < 1$. The linearized system in the vicinity of the point $y_2 = \frac{1-r}{r}$ is:

$$y(k+1) = -[r - 2 + 2ry(k)]_{\left(y=\frac{1-r}{r}\right)}\, y(k) = r\,y(k)$$

The equilibrium point $y_1 = \frac{1-r}{r}$ is stable for $0 < r < 1$, while it is unstable for $r > 1$.

1.b) When $r = 3$ the system becomes:

$$y(k+1) = -y(k) - 3\,y^2(k).$$

The stability of the equilibrium point $\mathbf{x} = 0$ is now studied using the function $V(y) = y^2 + \alpha y^3$. This functions is positive definite in $\mathbf{x} = 0$ for any value of parameter α. The function $\Delta V(y)$ is

$$\begin{aligned}
\Delta V(y) &= V(y(k+1)) - V(y(k)) \\
&= (-y - 3\,y^2)^2 + \alpha(-y - 3\,y^2)^3 - y^2 - \alpha y^3 \\
&= y^2 + 6y^3 + 9y^4 - \alpha(y^3 + 9y^4 + 27y^5 + 27y^6) - y^2 - \alpha y^3 \\
&= (6 - 2\alpha)y^3 + (9 - 9\alpha)y^4 - 27\alpha y^5 - 27\alpha y^6
\end{aligned}$$

If $\alpha = 3$ the function $\Delta V(y)$ is negative definite:

$$\Delta V(y) = -18y^4 - 81y^5 - 81y^6.$$

So the point $\mathbf{x} = 0$ is asymptotically stable for $r = 3$.

For $r = 1$ the two equilibrium points coincides: $y_1 = y_2 = 0$. In this case the difference equation of the system is:

$$y(k+1) = y(k) - y^2(k).$$

For studying the stability of the pointy $y = 0$ the following function can be used

$$V(y) = -y \qquad \rightarrow \qquad \Delta V(y) = -(y - y^2) - (-y) = y^2 > 0.$$

Since $\mathbf{x} = 0$ is an accumulation point for the set of all the point for which $V(y) > 0$, and being $\Delta V(y)$ positive definite, using the instability Lyapunov criterion one can conclude that $\mathbf{x} = 0$ is unstable.

So, the stability of the two equilibrium points for variable $r > 0$ is:

$$y_1 = 0 \;:\; \begin{cases} 1 < r \leq 3 & \text{as. stable} \\ r \leq 1 \;\text{ e }\; r > 3 & \text{unstable} \end{cases}$$

$$y_2 = \frac{1-r}{r} \;:\; \begin{cases} 1 < r & \text{as. stable} \\ r \geq 1 & \text{unstable} \end{cases}$$