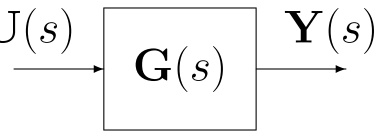
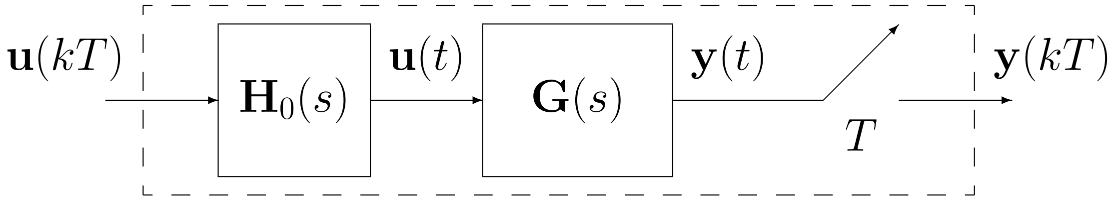
Sampled systems

- Let us consider the following continuous-time linear system:

$$\mathbf{G}(s): \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) \end{cases}$$

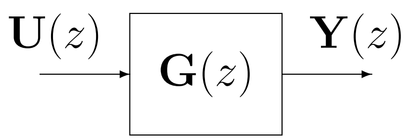


- Inserting a zero-order hold $\mathbf{H}_0(s)$ at the input and a sampler (of period T) at the output of system $\mathbf{G}(s)$, one obtains the following discrete system:



- The input-output dynamic behavior of the new sampled system $\mathbf{G}(z)$ is described by the following discrete time state space model:

$$\mathbf{G}(z): \quad \begin{cases} \mathbf{x}((k+1)T) = \mathbf{F}\,\mathbf{x}(kT) + \mathbf{G}\,\mathbf{u}(kT) \\ \mathbf{y}(kT) = \mathbf{H}\,\mathbf{x}(kT) \end{cases}$$



- Matrices $(\mathbf{A},\ \mathbf{B},\ \mathbf{C})$ and matrices $(\mathbf{F},\ \mathbf{G},\ \mathbf{H})$ are linked as follows:

$$\boxed{\mathbf{F} = e^{\mathbf{A}\,T}, \qquad \mathbf{G} = \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\,d\sigma, \qquad \mathbf{H} = \mathbf{C}}$$

- Being $\mathbf{F} = e^{\mathbf{A}T}$, the sampled system $\mathbf{G}(z)$ is stable if and only if the continuous-time system $\mathbf{G}(s)$ is stable.

- Since matrices $\mathbf{F}$ and $\mathbf{G}$ depend on parameter T, it is useful to study how the reachability and observability properties of the sampled system $\mathbf{G}(z)$ vary with the sampling period T.

- For a linear system with only one input, the following property holds.

  Theorem. *Let* $(\mathbf{A},\ \mathbf{b})$ *be a reachable system and let* T *be a sampling period. The corresponding sampled system is reachable if and only if for each couple* λ_i, λ_j *of different eigenvalues of matrix* $\mathbf{A}$ *having the same real part it is:*

$$\mathrm{Im}(\lambda_i - \lambda_j) \neq \frac{2k\pi}{T} = k\,\omega_s \qquad\qquad k = \pm 1,\ \pm 2,\ \pm 3,\ \ldots$$

- For linear system with only one output, the following property holds.

  Theorem. *Let* $(\mathbf{A},\ \mathbf{c})$ *be an observable system and let* T *be a sampling period. The corresponding sampled system is observable if and only if for each couple* λ_i, λ_j *of different eigenvalues of matrix* $\mathbf{A}$ *having the same real part it is:*

$$\mathrm{Im}(\lambda_i - \lambda_j) \neq \frac{2k\pi}{T} = k\,\omega_s \qquad\qquad k = \pm 1,\ \pm 2,\ \pm 3,\ \ldots$$

- A direct consequence of the previous two theorems is the following: if all the eigenvalues of matrix $\mathbf{A}$ are real, then the sampled system always holds, for each $T > 0$, the same structural properties (reachability and observability) of the corresponding continuous-time system $(\mathbf{A},\ \mathbf{b},\ \mathbf{c})$.

```
--- Matlab commands -----------------------------------------------------------------
A=[ -2 -3  0;           % System matrix
     3 -2  0;                                                          poles =
     1  1 -4];
b = [1; 0; 1];          % Input vector                                   -4.0000
c = [0  1  1];          % Output vector                                  -2.0000 + 3.0000i
Sys=ss(A,b,c,0);        % Continuous-time system                         -2.0000 - 3.0000i
poles=eig(A)            % Eigenvalues of the system
Rpiu=ctrb(A,b);         % Reachability matrix                          det_Rpiu =  42
det_Rpiu=det(Rpiu)      % Determinant of matrix Rpiu          -->
Omeno=obsv(A,c);        % Observability matrix                         det_Omeno = -75
det_Omeno=det(Omeno)    % Determinant of matrix Omeno
Tc=2*pi/6;              % Sampling period                              det_RpiuD =  0
SysD=c2d(Sys,Tc);       % Sampled system
[F,G,H]=ssdata(SysD);   % Matrices of the sampled system               det_OmenoD =  0
RpiuD=ctrb(F,G);        % Reachability matrix of the sampled system
det_RpiuD=det(RpiuD)    % Determinant of matrix RpiuD
OmenoD=obsv(F,H);       % Observability matrix of the sampled system
det_OmenoD=det(OmenoD)  % Determinant of matrix OmenoD
-------------------------------------------------------------------------------------
```

Example. Given the following continuous-time system:

$$\begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

compute the corresponding sampled system. Let:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Matrix $\mathbf{F}$ is:

$$\mathbf{F} = e^{\mathbf{A}T} = \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix}$$

Matrix $\mathbf{G}$ is:

$$\begin{aligned} \mathbf{G} &= \int_0^T e^{\mathbf{A}\sigma} \mathbf{B}\, d\,\sigma = \left(\int_0^T e^{\mathbf{A}\sigma}\, d\,\sigma \right) \mathbf{B} \\ &= \int_0^T \begin{bmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{bmatrix} d\,\sigma \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\sigma & -\cos\sigma \\ \cos\sigma & \sin\sigma \end{bmatrix} \Bigg|_0^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \sin T & 1-\cos T \\ \cos T - 1 & \sin T \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-\cos T \\ \sin T \end{bmatrix} \end{aligned}$$

Matrix $\mathbf{H}$ is equal to matrix $\mathbf{C}$.

The sampled system has the following structure:

$$\begin{cases} \mathbf{x}(k+1) &= \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1-\cos T \\ \sin T \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

where

$$\mathbf{x}(k) \equiv \mathbf{x}(kT), \qquad y(k) \equiv y(kT), \qquad u(k) \equiv u(kT)$$

The eigenvalues of matrix $\mathbf{A}$ are $\lambda_1 = j$, $\lambda_2 = -j$. The reachability matrix of the sampled system is:

$$\begin{aligned} \mathcal{R}^+ = [\mathbf{G}\ \ \mathbf{FG}] &= \begin{bmatrix} 1-\cos T & \cos T + \sin^2 T - \cos^2 T \\ \sin T & -\sin T + 2\sin T\cos T \end{bmatrix} \\ &= \begin{bmatrix} 1-\cos T & \cos T + \cos 2T \\ \sin T & -\sin T + \sin 2T \end{bmatrix} \end{aligned}$$

When $T = \pi$ the system is not completely reachable, in fact:

$$\mathcal{R}^+_{T=\pi} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

The determinant of matrix $\mathcal{R}^+$ is:

$$|\mathcal{R}^+| = -2\sin T(1-\cos T)$$

According to the reachability theorem, the sampled system is reachable if and only if:

$$T \neq \frac{2k\pi}{\text{Im}(\lambda_1-\lambda_2)} = \frac{2k\pi}{2} = k\pi$$

Similar considerations hold also for the observability matrix:

$$\mathcal{O}^- = \begin{bmatrix} 0 & 1 \\ -\sin T & \cos T \end{bmatrix}$$

Also in this case, the sampled system is observable if and only if $T \neq k\pi$. The characteristic polynomial of matrix $\mathbf{F}$ is:

$$|z\mathbf{I}-\mathbf{F}| = (z-\cos T)^2 + \sin^2 T$$

The eigenvalues of matrix $\mathbf{F}$ are the following:

$$z_{1,2} = \cos T \pm j\sin T = e^{\pm jT}$$

The transfer function $G(s)$ of the continuous-time system is:

$$G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s}{s^2+1}$$

The transfer function $G(z)$ of the corresponding sampled system is:

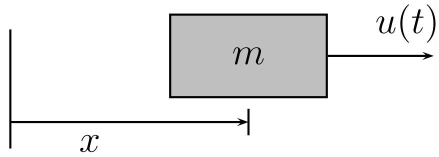
$$\begin{aligned}
G(z) &= \mathbf{H}(z\mathbf{I}-\mathbf{F})^{-1}\mathbf{G} \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z-\cos T & -\sin T \\ \sin T & z-\cos T \end{bmatrix}^{-1} \begin{bmatrix} 1-\cos T \\ \sin T \end{bmatrix} \\
&= \frac{1}{(z-\cos T)^2+\sin^2 T} \begin{bmatrix} -\sin T & z-\cos T \end{bmatrix} \begin{bmatrix} 1-\cos T \\ \sin T \end{bmatrix} \\
&= \frac{\sin T\cos T - \sin T + z\sin T - \sin T\cos T}{z^2 - 2\cos T z + 1} \\
&= \frac{\sin T(z-1)}{z^2-2\cos T z+1}
\end{aligned}$$

The same result can also be obtained applying the $\mathcal{Z}$-transform to the function $H(s)G(s)$ where $H(s)$ is the zero order hold:

$$\begin{aligned}
G(z) &= \mathcal{Z}\left[H_0(s)G(s)\right] = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\frac{s}{s^2+1}\right] = (1-z^{-1})\mathcal{Z}\left[\frac{1}{(s^2+1)}\right] \\
&= (1-z^{-1})\frac{z\sin T}{z^2-2\cos T z+1} = \frac{\sin T(z-1)}{z^2-2\cos T z+1}
\end{aligned}$$

Example. Let us consider the following inertial system composed by an external force $u(t)$ acting on a mass ($m = 1$):

$$m\ddot{x} = u(t)$$



Let us compute a discrete static state feedback $u(k) = \mathbf{K}\mathbf{x}(k)$ which puts in the origin the two eigenvalues of the feedback system (dead-beat controller).

Let $\mathbf{x}$ be the state vector:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x_1 = x, \qquad x_2 = \dot{x}$$

The state space equations of the system are the following:

$$\begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

The matrices $\mathbf{F}$ and $\mathbf{G}$ of the corresponding sampled system are:

$$\mathbf{F} = e^{\mathbf{A}T} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbf{G} = \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\, d\sigma = \int_0^T \begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\sigma = \int_0^T \begin{bmatrix} \sigma \\ 1 \end{bmatrix} d\sigma = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$$

The sampled system has the following structure:

$$\begin{cases} \mathbf{x}(k+1) &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

The characteristic polynomial of matrix $\mathbf{F}$ is:

$$\Delta_{\mathbf{F}}(z) = (z-1)^2 = z^2 - 2z + 1.$$

Let $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^T$ be the feedback gain vector. Setting $u(k) = \mathbf{K}\mathbf{x}(k)$, one obtains the following system matrix for the discrete feedback system:

$$\mathbf{F} + \mathbf{G}\mathbf{K} = \begin{bmatrix} 1 + k_1\frac{T^2}{2} & T + k_2\frac{T^2}{2} \\ k_1 T & 1 + k_2 T \end{bmatrix}$$

The characteristic polynomial of matrix $\mathbf{F} + \mathbf{G}\mathbf{K}$ is:

$$\begin{aligned} \lambda_{\mathbf{F}+\mathbf{G}\mathbf{K}} &= (z - 1 - k_1\tfrac{T^2}{2})(z - 1 - k_2 T) - k_1 T(T + k_2\tfrac{T^2}{2}) \\ &= z^2 - (2 + k_2 T + k_1\tfrac{T^2}{2})z + (1 + k_2 T - k_1\tfrac{T^2}{2}) \end{aligned}$$

Imposing $\lambda_{\mathbf{F}+\mathbf{GK}} = z^2$ one obtains:

$$\left\{\begin{array}{l} 2+k_2T+k_1\frac{T^2}{2}=0 \\ 1+k_2T-k_1\frac{T^2}{2}=0 \end{array}\right. \quad \rightarrow \quad \left\{\begin{array}{l} 3+2k_2T=0 \\ 1+k_1T^2=0 \end{array}\right. \quad \rightarrow \quad \left\{\begin{array}{l} k_2=-\frac{3}{2T} \\ k_1=-\frac{1}{T^2} \end{array}\right.$$

that is:

$$\mathbf{K} = \left[\begin{array}{cc} -\frac{1}{T^2} & -\frac{3}{2T} \end{array}\right]$$

The same result can also be obtained as follows:

$$\begin{aligned}
\mathbf{K} &= \left[\begin{array}{cc} 1 & -2 \end{array}\right] \left\{ \left[\begin{array}{cc} \frac{T^2}{2} & \frac{3}{2}T^2 \\ T & T \end{array}\right] \left[\begin{array}{cc} -2 & 1 \\ 1 & 0 \end{array}\right] \right\}^{-1} \\
&= \left[\begin{array}{cc} 1 & -2 \end{array}\right] \left[\begin{array}{cc} \frac{T^2}{2} & \frac{T^2}{2} \\ -T & T \end{array}\right]^{-1} \\
&= \left[\begin{array}{cc} 1 & -2 \end{array}\right] \frac{1}{T^3} \left[\begin{array}{cc} T & -\frac{T^2}{2} \\ T & \frac{T^2}{2} \end{array}\right] \\
&= \left[\begin{array}{cc} -\frac{1}{T^2} & -\frac{3}{2T} \end{array}\right]
\end{aligned}$$

Substituting, matrix $\mathbf{F}+\mathbf{GK}$ can be expressed as follows:

$$\mathbf{F}+\mathbf{GK} = \left[\begin{array}{cc} \frac{1}{2} & \frac{T}{4} \\ -\frac{1}{T} & -\frac{1}{2} \end{array}\right]$$

A dead-beat controller always satisfies the following property: the system state $\mathbf{x}$ is moved from the initial point $\mathbf{x}(0)$ to the final zero state $\mathbf{x}=0$ in a number n of steps which is equal to the dimension of state space (in this case $n=2$), whatever it is the value of the sampling period T.
Clearly, when the sampling period T decreases the amplitude of the control signal $u(k)$ increases. In fact, it is:

$$u(0) = \mathbf{K}\mathbf{x}(0) = \left[\begin{array}{cc} -\frac{1}{T^2} & -\frac{3}{2T} \end{array}\right] \mathbf{x}(0)$$

and

$$\begin{aligned}
u(1) &= \mathbf{K}\mathbf{x}(1) = \mathbf{K}(\mathbf{F}+\mathbf{GK})\mathbf{x}(0) \\
&= \left[\begin{array}{cc} -\frac{1}{T^2} & -\frac{3}{2T} \end{array}\right] \left[\begin{array}{cc} \frac{1}{2} & \frac{T}{4} \\ -\frac{1}{T} & -\frac{1}{2} \end{array}\right] \mathbf{x}(0) \\
&= \left[\begin{array}{cc} \frac{1}{T^2} & \frac{1}{2T} \end{array}\right] \mathbf{x}(0)
\end{aligned}$$

If the state $x_2 = \dot{x}$ is not measurable, a *dead-beat reduced order observer* can be used. The state space transformation $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ transforms the system as follows

$$\mathbf{P} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] = \mathbf{P}^{-1} \quad \rightarrow \quad \left\{\begin{array}{rcl} \bar{\mathbf{x}}(k+1) &=& \left[\begin{array}{cc} 1 & 0 \\ T & 1 \end{array}\right] \bar{\mathbf{x}}(k) + \left[\begin{array}{c} T \\ \frac{T^2}{2} \end{array}\right] u(k) \\ y(k) &=& \left[\begin{array}{cc} 0 & 1 \end{array}\right] \bar{\mathbf{x}}(k) \end{array}\right.$$

From relation $\mathbf{A}_{11} + \mathbf{L}\mathbf{A}_{21} = 1 + \mathbf{L}T = 0$ one obtains $\mathbf{L} = -\frac{1}{T}$. The dead beat reduced order observer has the following form:

$$\hat{\mathbf{x}}(k) = \mathbf{P} \begin{bmatrix} \hat{v}(k) - \mathbf{L}y(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} y(k) \\ \hat{v}(k) + \frac{y(k)}{T} \end{bmatrix}$$

where

$$\hat{v}(k+1) = (\mathbf{A}_{12} + \mathbf{L}\mathbf{A}_{22})y(k) + (\mathbf{B}_1 + \mathbf{L}\mathbf{B}_2)u(k) = -\tfrac{1}{T}y(k) + \tfrac{T}{2}u(k)$$

The transfer function $G(s)$ of the continuous-time system is:

$$G(s) = \frac{1}{s^2}$$

The transfer function $G(z)$ of the corresponding sampled system is

$$\begin{aligned} G(z) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-1 & -T \\ 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} \\ &= \tfrac{1}{(z-1)^2} \begin{bmatrix} z-1 & T \end{bmatrix} \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} \\ &= \frac{T^2}{2} \frac{z+1}{(z-1)^2} \end{aligned}$$

The sampled system $g(z)$ can be obtained as follows:

$$\begin{aligned} G(z) &= \mathcal{Z}\left[H_0(s)G(s)\right] \\ &= \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\frac{1}{s^2}\right] = (1-z^{-1})\mathcal{Z}\left[\frac{1}{s^3}\right] \\ &= (1-z^{-1})\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3} \\ &= \frac{T^2}{2}\frac{(z+1)}{(z-1)^2} \end{aligned}$$

Example. Let us discretize the following continuous-time linearized model of an inverted pendulum:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ \alpha g & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ -\frac{\alpha}{M} \end{bmatrix} \mathbf{u}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{C}\,\mathbf{x}(t)$$

The transfer function $G(s)$ of the system is:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{-\lambda^2}{gM(s-\lambda)(s+\lambda)}$$

where $\lambda = \sqrt{\alpha g}$. The corresponding discrete sampled system $(\mathbf{F},\ \mathbf{G},\ \mathbf{H})$ can be determined as follows. Let T the sampling period. The eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of matrix $\mathbf{A}$ corresponding to the two eigenvalues $\lambda_{1,2} = \pm\lambda$ are:

$$\lambda_1 = -\lambda \quad \rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -\lambda \end{bmatrix}, \qquad \lambda_2 = \lambda \quad \rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$$

Setting $\mathbf{T} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$, matrix $\mathbf{F}$ can be computed as follows:

$$\begin{aligned}
\mathbf{F} &= e^{\mathbf{A}T} = \mathbf{T}e^{\bar{\mathbf{A}}}\mathbf{T}^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -\lambda & \lambda \end{bmatrix}\begin{bmatrix} e^{-\lambda T} & 0 \\ 0 & e^{\lambda T} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{\lambda} \\ 1 & \frac{1}{\lambda} \end{bmatrix} \\
&= \tfrac{1}{2}\begin{bmatrix} e^{\lambda T} + e^{-\lambda T} & \dfrac{e^{\lambda T} - e^{-\lambda T}}{\lambda} \\ \lambda(e^{\lambda T} - e^{-\lambda T}) & e^{\lambda T} + e^{-\lambda T} \end{bmatrix} = \begin{bmatrix} \cosh(\lambda T) & \dfrac{\sinh(\lambda T)}{\lambda} \\ \lambda\sinh(\lambda T) & \cosh(\lambda T) \end{bmatrix}
\end{aligned}$$

Matrix $\mathbf{G}$ has the following form:

$$\begin{aligned}
\mathbf{G} &= \int_0^T e^{\mathbf{A}\sigma}\mathbf{B}\,d\sigma = \int_0^T \begin{bmatrix} \cosh(\lambda T) & \frac{\sinh(\lambda T)}{\lambda} \\ \lambda\sinh(\lambda T) & \cosh(\lambda T) \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{\lambda^2}{gM} \end{bmatrix} d\sigma \\
&= \frac{1}{gM}\int_0^T \begin{bmatrix} -\lambda\sinh(\lambda\sigma) \\ -\lambda^2\cosh(\lambda\sigma) \end{bmatrix} d\sigma = \frac{1}{gM}\begin{bmatrix} -\cosh(\lambda\sigma) \\ -\lambda\sinh(\lambda\sigma) \end{bmatrix}_0^T \\
&= \frac{1}{gM}\begin{bmatrix} 1 - \cosh(\lambda T) \\ -\lambda\sinh(\lambda T) \end{bmatrix}
\end{aligned}$$

Since $\mathbf{H} = \mathbf{C}$, the sampled system $(\mathbf{F},\ \mathbf{G},\ \mathbf{H})$ has the following structure:

$$\begin{cases}
\mathbf{x}(k+1) = \begin{bmatrix} \cosh(\lambda T) & \frac{\sinh(\lambda T)}{\lambda} \\ \lambda\sinh(\lambda T) & \cosh(\lambda T) \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} \frac{1-\cosh(\lambda T)}{gM} \\ -\frac{\lambda\sinh(\lambda T)}{gM} \end{bmatrix}\mathbf{u}(k) \\
y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(k)
\end{cases}$$

The system is reachable and observable for any value of $T > 0$. In fact, the reachability matrix:

$$\mathcal{R}^+ = \frac{1}{gM}\begin{bmatrix} 1-\cosh(\lambda T) & \cosh(\lambda T)-\cosh(2\lambda T) \\ \lambda\sinh(\lambda T) & \lambda[1-2\cosh(\lambda T)]\sinh(\lambda T)\end{bmatrix}$$

has the following determinant:

$$\det\mathcal{R}^+ = -2\lambda\frac{[\cosh(\lambda T)-1]\sinh(\lambda T)}{g^2M^2}$$

which is zero only when $T = 0$. Moreover, the observability matrix:

$$\mathcal{O}^- = \begin{bmatrix} 1 & 0 \\ \cosh(\lambda T) & \frac{\sinh(\lambda T)}{\lambda}\end{bmatrix}$$

has a zero determinant only when $T = 0$. It can be easily proved that the transfer function $G(z)$ of the discrete system is:

$$G(z) = \mathbf{H}(z\mathbf{I}-\mathbf{F})^{-1}\mathbf{G} = \frac{(z+1)[1-\cosh(\lambda T)]}{gM(z-e^{\lambda T})(z-e^{-\lambda T})}$$

One can easily verify that the transfer function $G(z)$ can also be obtained using the following formula:

$$G(z) = \mathcal{Z}\left[H_0(s)G(s)\right] = \mathcal{Z}\left[\frac{1-e^{sT}}{s}G(s)\right] = (1-z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right]$$

where $G(s)$ is the transfer function of the continuous-time system:

$$G(s) = \frac{-\lambda^2}{gM(s-\lambda)(s+\lambda)}.$$