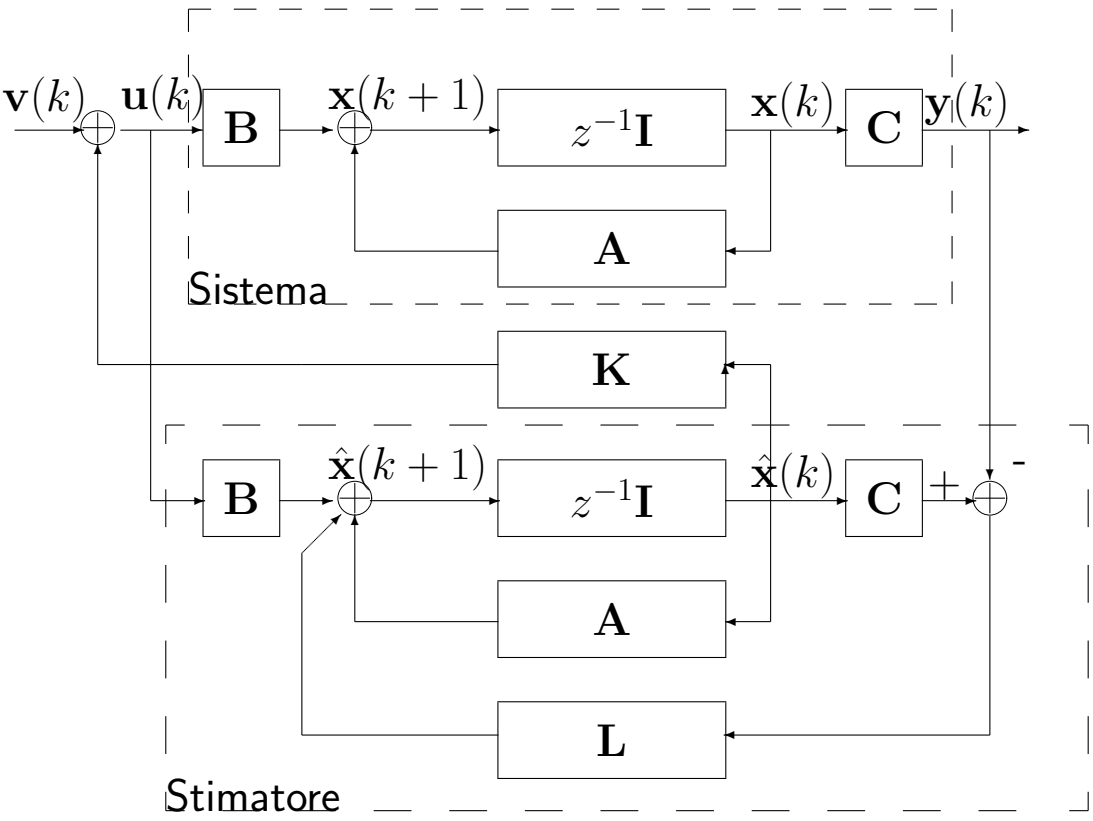
Regulator design

- The *regulator* is the system composed by the state observer and the static state feedback $\mathbf{u} = \mathbf{K}\mathbf{x}$. For a discrete system it is:



- State space equations of the global feedback system:

$$\text{System:} \quad \begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \end{cases}$$

$$\text{Feedback:} \quad \mathbf{u}(k) = \mathbf{K}\hat{\mathbf{x}}(k) + \mathbf{v}(k)$$

$$\text{Estimator:} \quad \hat{\mathbf{x}}(k+1) = \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{L}[\mathbf{C}\hat{\mathbf{x}}(k) - \mathbf{y}(k)]$$

- Substituting the control signal $\mathbf{u}(k)$ one obtains:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{v}(k) \\ \hat{\mathbf{x}}(k+1) = (\mathbf{A} + \mathbf{L}\mathbf{C} + \mathbf{B}\mathbf{K})\hat{\mathbf{x}}(k) - \mathbf{L}\mathbf{C}\mathbf{x}(k) + \mathbf{B}\mathbf{v}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \end{cases}$$

- The obtained system $(\overline{\mathbf{A}},\ \overline{\mathbf{B}},\ \overline{\mathbf{C}})$ has dimension $2n$:

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \hat{\mathbf{x}}(k+1) \end{bmatrix} = \overbrace{\begin{bmatrix} \mathbf{A} & \mathbf{BK} \\ -\mathbf{LC} & \mathbf{A}+\mathbf{LC}+\mathbf{BK} \end{bmatrix}}^{\overline{\mathbf{A}}} \begin{bmatrix} \mathbf{x}(k) \\ \hat{\mathbf{x}}(k) \end{bmatrix} + \overbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix}}^{\overline{\mathbf{B}}} \mathbf{v}(k)$$

$$\mathbf{y}(k) = \overbrace{\begin{bmatrix} \mathbf{C} & 0 \end{bmatrix}}^{\overline{\mathbf{C}}} \begin{bmatrix} \mathbf{x}(k) \\ \hat{\mathbf{x}}(k) \end{bmatrix}$$

- The structural properties of the obtained system can be shown using the following state space transformation:

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}, \qquad \mathbf{T}^{-1} = \mathbf{T}$$

- The second part of vector $\overline{\mathbf{x}}$ is equal to the estimation error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$:

$$\overline{\mathbf{x}} = \mathbf{T}^{-1} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x} - \hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

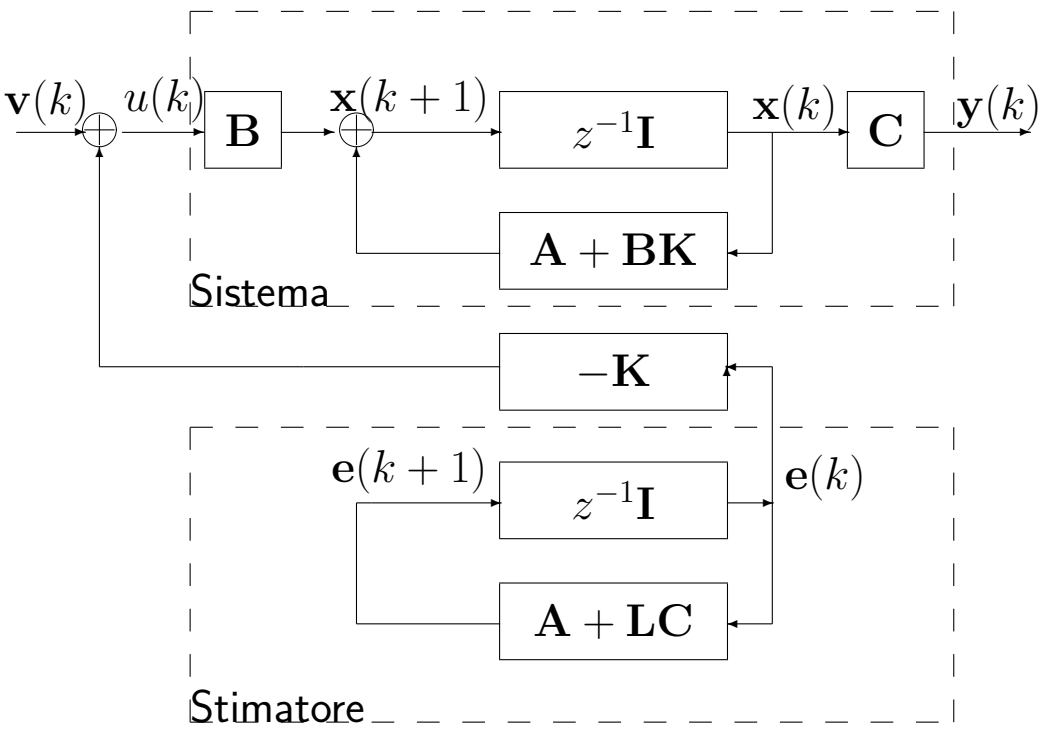
- The transformed system ($\mathcal{A} = \mathbf{T}^{-1}\overline{\mathbf{A}}\mathbf{T}$, $\mathcal{B} = \mathbf{T}^{-1}\overline{\mathbf{B}}$ e $\mathcal{C} = \overline{\mathbf{C}}\mathbf{T}$) is:

$$\begin{cases} \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{e}(k+1) \end{bmatrix} = \overbrace{\begin{bmatrix} \mathbf{A}+\mathbf{BK} & -\mathbf{BK} \\ \mathbf{0} & \mathbf{A}+\mathbf{LC} \end{bmatrix}}^{\mathcal{A}} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix} + \overbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}}^{\mathcal{B}} \mathbf{v}(k) \\ \mathbf{y}(k) = \underbrace{\begin{bmatrix} \mathbf{C} & 0 \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix} \end{cases}$$

- The system is in the reachability standard form. The system eigenvalues are the union of the eigenvalues of the two matrices $\mathbf{A}+\mathbf{BK}$ and $\mathbf{A}+\mathbf{LC}$.

- *Separation property*. The following two problems can be designed independently:
  1) design the *"static state feedback"*, that is the placement of the eigenvalues of matrix $(\mathbf{A}+\mathbf{BK})$;
  2) design the *"state observer"*, that is the placement of the eigenvalues of matrix $(\mathbf{A}+\mathbf{LC})$

- The transformed system can be graphically represented as follow:



- The part of the transformed system which describes the observer is an autonomous subsystem without inputs. Its output $\mathbf{e}(k)$ (the estimation error) acts as an input of the reachable part of the feedback dynamic system.

- The presence of the state estimator does not modify the *input–output* relationship of the considered system. In fact, the transfer matrix $\mathbf{H}(z)$ of the system remains the same in the following two cases: the "real" state $\mathbf{x}$ is or the "estimate" state $\hat{\mathbf{x}}$ is used.

$$\begin{aligned}
\mathbf{H}(z) &= \overline{\mathbf{C}}(z\mathbf{I}-\overline{\mathbf{A}})^{-1}\overline{\mathbf{B}} = \mathcal{C}(z\mathbf{I}-\mathcal{A})^{-1}\mathcal{B} \\
&= \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} [z\mathbf{I}-(\mathbf{A}+\mathbf{BK})]^{-1} & * * * * * * \\ 0 & [z\mathbf{I}-(\mathbf{A}+\mathbf{LC})]^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \\
&= \mathbf{C}[z\mathbf{I}-(\mathbf{A}+\mathbf{BK})]^{-1}\mathbf{B}
\end{aligned}$$