Recalls of Automatic Control

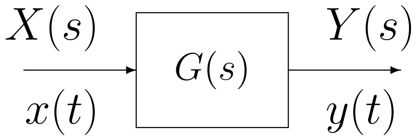
- The physical systems are often mathematically described by linear time-invariant differential equations. Example:

$$\ddot{y}(t) + 3\,\dot{y}(t) + 2\,y(t) = \dot{x}(t) + 5\,x(t)$$

- In the Laplace transformed space the solution $Y(s)$ of the differential equation, when the initial conditions are zero, is the product of the transfer function $G(s)$ and the Laplace transform $X(s)$ of the input signal $x(t)$:

$$Y(s) = \underbrace{\frac{(s+5)}{s^2 + 3\,s + 2}}_{G(s)} X(s) = G(s)\,X(s)$$

- The system can be graphically described by a SISO (Single Input Single Output) block scheme:



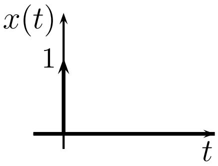
- The time solution $y(t)$ of the given differential equation can be obtained using the inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)X(s)]$$

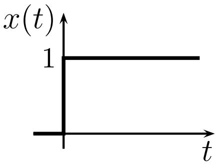
- The Laplace approach:
  - is powerful and quite simple, but **only the input and output signals $x(t)$ and $y(t)$ of the given system are considered.**
  - **does not show** the time behavior of **the internal variables of the system.**
  - **is not suitable for describing some "particular" linear systems.**
  - **is not suitable for describing nonlinear systems.**

- Laplace transform of the main time signals:

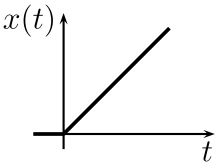
$$x(t) = \delta(t) \quad \leftrightarrow \quad X(s) = 1$$



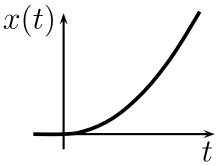
$$x(t) = u(t) \quad \leftrightarrow \quad X(s) = \frac{1}{s}$$



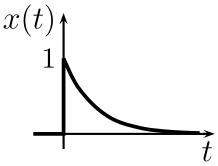
$$x(t) = t \quad \leftrightarrow \quad X(s) = \frac{1}{s^2}$$



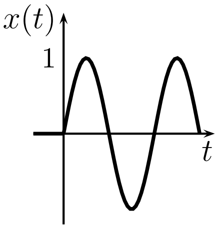
$$x(t) = \frac{t^2}{2} \quad \leftrightarrow \quad X(s) = \frac{1}{s^3}$$



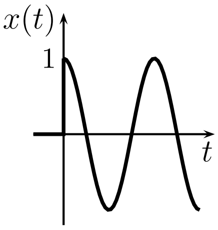
$$x(t) = e^{-a\,t} \quad \leftrightarrow \quad X(s) = \frac{1}{s+a}$$



$$x(t) = \sin \omega t \quad \leftrightarrow \quad X(s) = \frac{\omega}{s^2+\omega^2}$$



$$x(t) = \cos \omega t \quad \leftrightarrow \quad X(s) = \frac{s}{s^2+\omega^2}$$



$$x(t) = t^n e^{-a\,t} \quad \leftrightarrow \quad X(s) = \frac{n!}{(s+a)^{n+1}}$$

$$x(t) = e^{-a\,t} \sin \omega t \quad \leftrightarrow \quad X(s) = \frac{\omega}{(s+a)^2+\omega^2}$$

$$x(t) = e^{-a\,t} \cos \omega t \quad \leftrightarrow \quad X(s) = \frac{s+a}{(s+a)^2+\omega^2}$$

- Main properties of the Laplace transform:

  Linearity:

$$\mathcal{L}\big[c_1\, x_1(t) + c_2\, x_2(t)\big] = c_1\, X_1(s) + c_2\, X_2(s)$$

  Time shift:

$$\mathcal{L}\big[x(t - t_0)\big] = e^{-t_0\, s}\, X(s)$$

  Time integral:

$$\mathcal{L}\left[\int_0^t x(\tau)\, d\tau\right] = \frac{1}{s}\, X(s)$$

  Time derivative (x_0 is the initial condition):

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = s\, X(s) - x_0$$

  Initial value theorem:

$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} s\, X(s)$$

  Final value theorem (only if $X(s)$ is a stable function):

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s\, X(s)$$

  Multiplying an exponential function:

$$\mathcal{L}\left[e^{-at} x(t)\right] = X(s + a)$$

- Examples of Laplace transforms in Matlab:

```
-------------------------------------------------------------------------
   syms s t                                      5        15
   xt=5+3*exp(-2*t)*sin(5*t);                    - + -------------
   Xs=laplace(xt,t,s);            -->            s          2
   pretty(Xs)                                        (s + 2)  + 25
-------------------------------------------------------------------------
```

```
-------------------------------------------------------------------------
   syms s t                                           8         4 s
   xt=4*cos(5*t)+4*t^2*exp(-3*t);                  -------- + -------
   Xs=laplace(xt,t,s);                 -->                3    2
   pretty(Xs)                                      (s + 3)    s  + 25
-------------------------------------------------------------------------
```

Partial fraction expansion method

- A rational function $Y(s)$ which has only *simple poles* can always be expressed as follows:

$$Y(s) = \frac{N(s)}{(s-p_1)\,(s-p_2)\ldots(s-p_n)} = \sum_{i=1}^{n} \frac{K_i}{s-p_i}$$

where parameters K_i (called $residues$)can be obtained as follows:

$$K_i = (s-p_i)Y(s)\Big|_{s=p_i}$$

- So, the time function $y(t)$ can be expressed as follows:

$$y(t) = \sum_{i=1}^{n} K_i\, e^{p_i t}$$

- Examples of inverse Laplace transforms in Matlab:

```
-------------------------------------------------------------------------
 syms s t
 Xs=(5*s+3)/((s+1)*(s+2)*(s+3));
 xt=ilaplace(Xs,s,t);            -->   7 exp(-2 t) - exp(-t) - 6 exp(-3 t)
 pretty(xt)
-------------------------------------------------------------------------
```

```
-------------------------------------------------------------------------
 syms s t
 Xs=(7*s^2-8*s+5)/(s^3+2*s^2+5*s);                 /            4 sin(2 t) \
 xt=ilaplace(Xs,s,t);              -->   6 exp(-t) | cos(2 t) - ---------- | + 1
 pretty(xt)                                        \                3      /
-------------------------------------------------------------------------
```

```
-------------------------------------------------------------------------
 syms s t                         2
 Xs=(5)/(s^3*(s+1)^2);         5 t
 xt=ilaplace(Xs,s,t);   -->    ---- - 15 exp(-t) - 5 t exp(-t) - 10 t + 15
 pretty(xt)                     2
-------------------------------------------------------------------------
```

Step Response of First Order Systems

- Let us consider a linear first order differential equation:

$$a\,\dot{y}(t) + b\,y(t) = c\,x(t)$$

- The corresponding transfer function $G(s)$ is:

$$G(s) = \frac{c}{a\,s+b} = \frac{K_0}{(1+\tau\,s)}$$

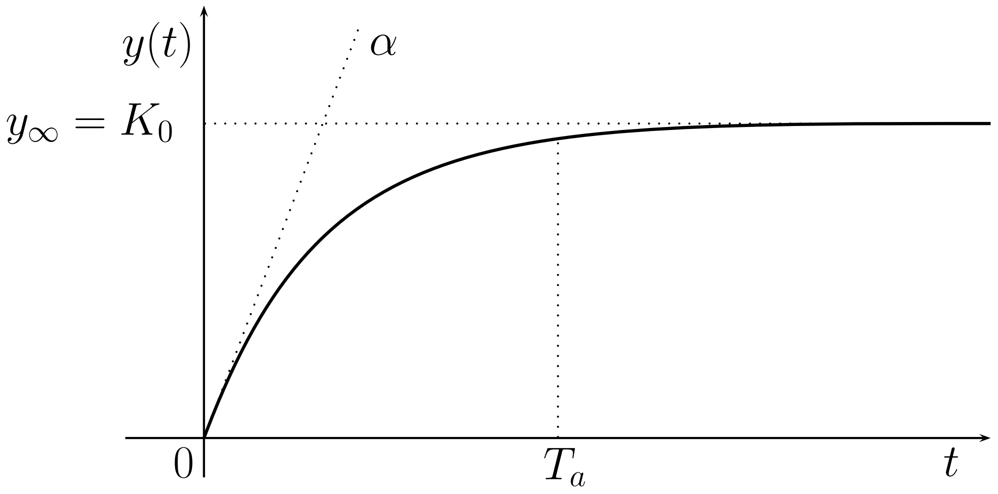
- The *constant time* τ and the *static gain* K_0 are defined as follows:

$$\tau = \frac{a}{b}, \qquad\qquad K_0 = G(s)|_{s=0} = G(0) = \frac{c}{b}.$$

- When $x(t) = 1$, the step response $y(t)$ of system $G(s)$ is:

$$y(t) = \mathcal{L}^{-1}\left[G(s)\,\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{K_0}{s(1+\tau\,s)}\right] = K_0\left(1-e^{-\frac{t}{\tau}}\right)$$

- The time behavior of function $y(t)$ is aperiodic and exponential:



- The *settling time* T_a is:

$$T_a = 3\tau = \frac{3}{|p|} = \frac{3}{\delta\omega_n}, \qquad\qquad p = -\frac{1}{\tau}$$

- When $x(t) = 1$, the *final value* y_∞ is equal to the static gain K_0:

$$y_\infty = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,Y(s) = K_0 = \frac{c}{b}$$

- The *initial slope* α can be determined using the initial value theorem:

$$\alpha = \lim_{t\to 0^+} \dot{y}(t) = \lim_{s\to\infty} s(sY(s)) = \lim_{s\to\infty} sG(s) = \frac{c}{a}$$

Step Response of Second Order Systems

- Let us consider a linear second order differential equation:

$$a\,\ddot{y}(t) + b\,\dot{y}(t) + c\,y(t) = d\,x(t)$$

- The corresponding transfer function $G(s)$ is:

$$G(s) = \frac{d}{a\,s^2 + b\,s + c} = \frac{K_0}{\dfrac{s^2}{\omega_n^2} + 2\delta\dfrac{s}{\omega_n} + 1} = \frac{K_0\,\omega_n^2}{s^2 + 2\delta\omega_n\,s + \omega_n^2}.$$

- The *natural frequency* ω_n, the *dumping ratio* δ and the *static gain* K_0 are defined as follows:

$$\omega_n = \sqrt{\frac{c}{a}}, \qquad \delta = \frac{b}{2\sqrt{ac}}, \qquad K_0 = \frac{d}{c}.$$

- The step response $y(t)$ of system $G(s)$ is:

$$y(t) = \mathcal{L}^{-1}\left[\frac{K_0\,\omega_n^2}{s(s^2 + 2\delta\omega_n\,s + \omega_n^2)}\right] = K_0\left[1 - \frac{e^{-\sigma t}}{\sqrt{1-\delta^2}}\,\mathrm{sen}\,(\omega\,t + \varphi)\right].$$

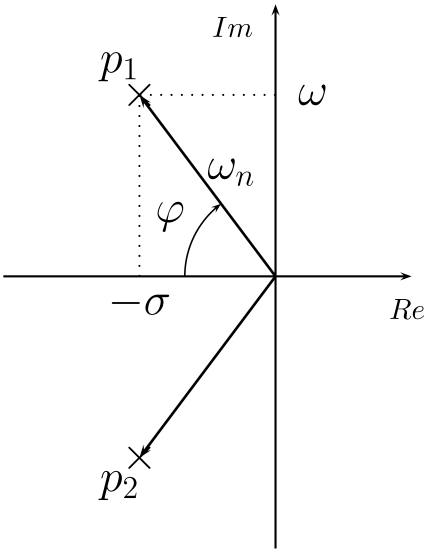
- The parameters $-\sigma$ and ω are the real and imaginary parts of the two complex coniugate poles $p_{1,2}$ of function $G(s)$:

$$p_{1,2} = -\sigma \pm j\omega$$

$$\sigma = \delta\omega_n = \tfrac{b}{2a}$$

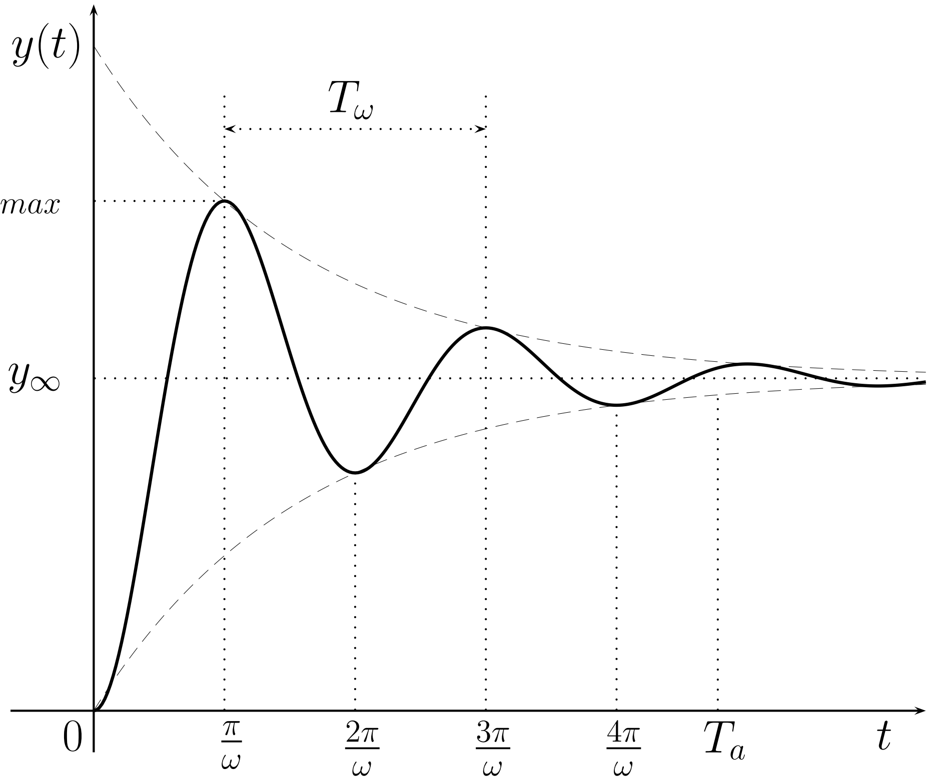
$$\omega = \omega_n\sqrt{1-\delta^2} = \tfrac{\sqrt{4ac-b^2}}{2a}$$

$$\varphi = \arccos\delta = \arctan\tfrac{\sqrt{1-\delta^2}}{\delta}$$



- This time analysis is valid only for $|\delta| \le 1$. When $0 \le \delta \le 1$ the system is stable: $\sigma \ge 0$. When $-1 \le \delta < 0$ the system is unstable: $\sigma < 0$.

- The time behavior of function $y(t)$ is exponential dumped sinusoidal:



- The *settling time* T_a can be determined ad follows:

$$T_a = \frac{3}{\sigma} = \frac{3}{\delta\omega_n},$$

- The *period* T_ω can be determined ad follows:

$$T_\omega = \frac{2\pi}{\omega}$$

- The *percent overshoot* $S\%$ is defined as follows:

$$S\% = 100\,\frac{(y_{max} - y_\infty)}{y_\infty}$$

- The *percent overshoot* $S\%$ can be determined as follows:

$$S\% = 100\; e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}} = 100\; e^{\frac{-\pi}{\tan\varphi}} = 100\; e^{\frac{-\pi\omega}{\sigma}}$$

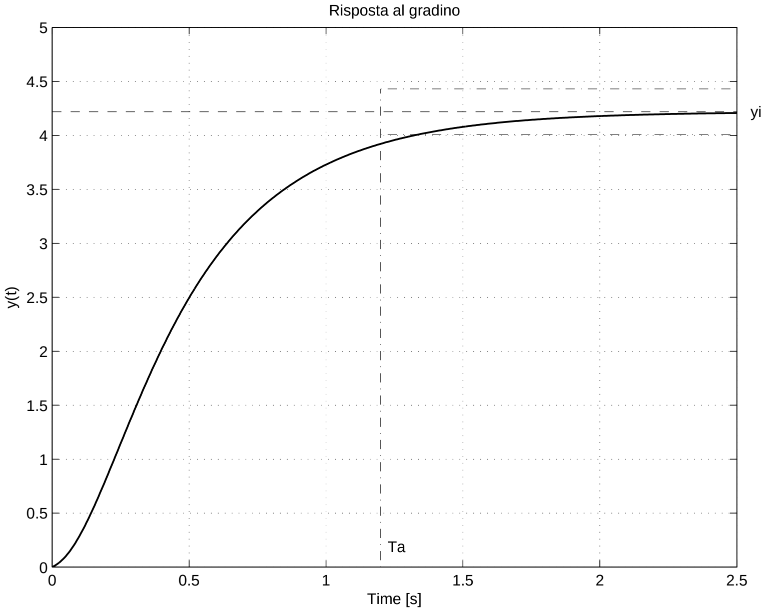
Step response of systems with one dominant pole

- Given the following system $G(s)$:

$$G(s) = \frac{(3+0.2\,s)(s^2+60\,s+1800)}{\underbrace{(2+0.8\,s)}_{\text{polo dominante}}(8+0.2\,s)(s^2+16\,s+80)}$$

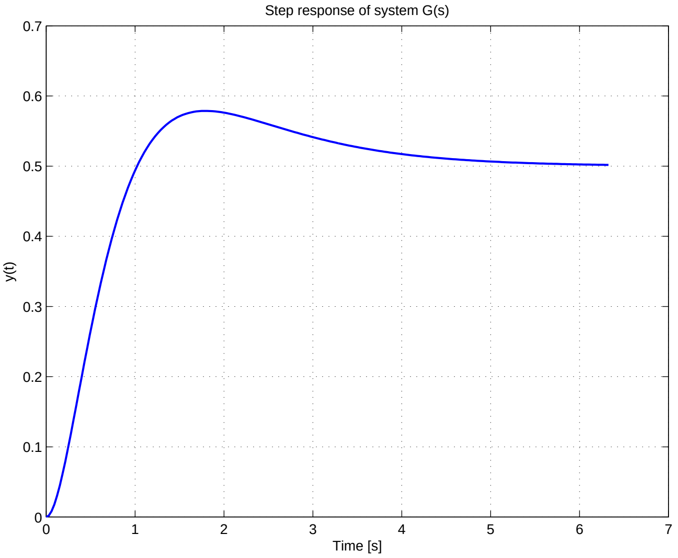
- Dominant pole p_1, final value y_∞, settling time T_a and time behavior of function $G(s)$:

$$\begin{aligned} p_1 &= G(0) = -2.5 \\ y_\infty &= G(0) = 4.22 \\ T_a &\simeq \frac{3}{2.5}\ \text{s} = 1.2\ \text{s}, \end{aligned}$$



- Examples of step response in Matlab:

```
----------------------------------------
 s=tf('s');
 Gs=(5*s+3)/((s+1)*(s+2)*(s+3));
 [yt,t]=step(Gs);
 figure(1); clf
 plot(t,yt,'Linewidth',1.5);
 grid on
 title('Step response of system G(s)')
 ylabel('y(t)')
 xlabel('Time [s]')
 --------------------------------------
```



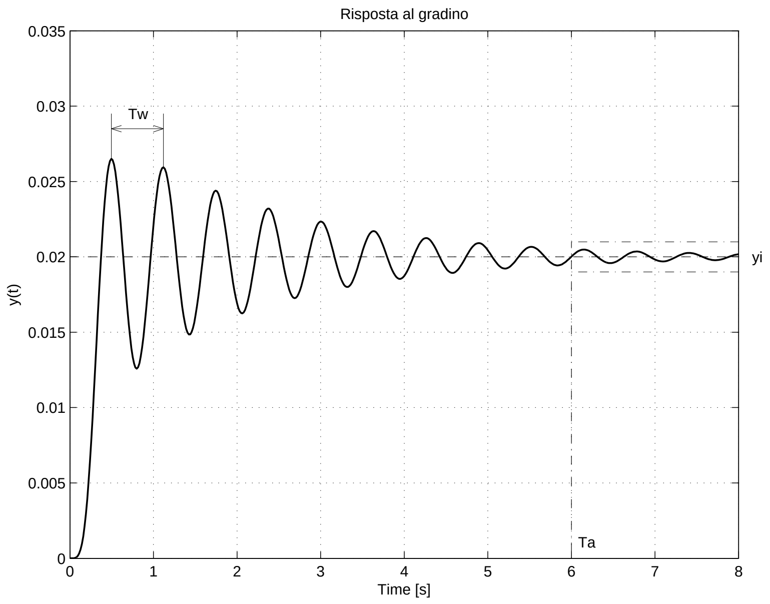
Step response of systems with two dominant poles

- Given the following system $G(s)$:

$$G(s) = \frac{800\,(2\,s+30)}{(0.2\,s+3)(2\,s+10)\,\underbrace{(s^2+s+100)}_{\text{poli dominanti}}(s^2+20s+400)}$$

- Dominant poles $p_{1,2}$, final value y_∞, settling time T_a, period T_ω and time behavior of function $G(s)$:

$$\begin{aligned} p_{1,2} &\simeq -0.5 \pm j\,10 \\ y_\infty &= G(0) = 0.02. \\ T_a &\simeq \frac{3}{0.5}\ \text{s} = 6\ \text{s} \\ T_\omega &\simeq \frac{2\pi}{10}\ \text{s} = 0.63\ \text{s} \end{aligned}$$



- Examples of step response in Matlab:

```
----------------------------------------
 s=tf('s');
 Gs=(5*s+30)/((s^2+2*s+15)*(0.2*s+4)*(0.1*s+2));
 [yt,t]=step(Gs,5);
 figure(1); clf
 plot(t,yt,'Linewidth',1.5);
 grid on
 title('Step response of system G(s)')
 ylabel('y(t)')
 xlabel('Time [s]')
 print -depsc Step_Gs_P12.eps
 ---------------------------------------
```