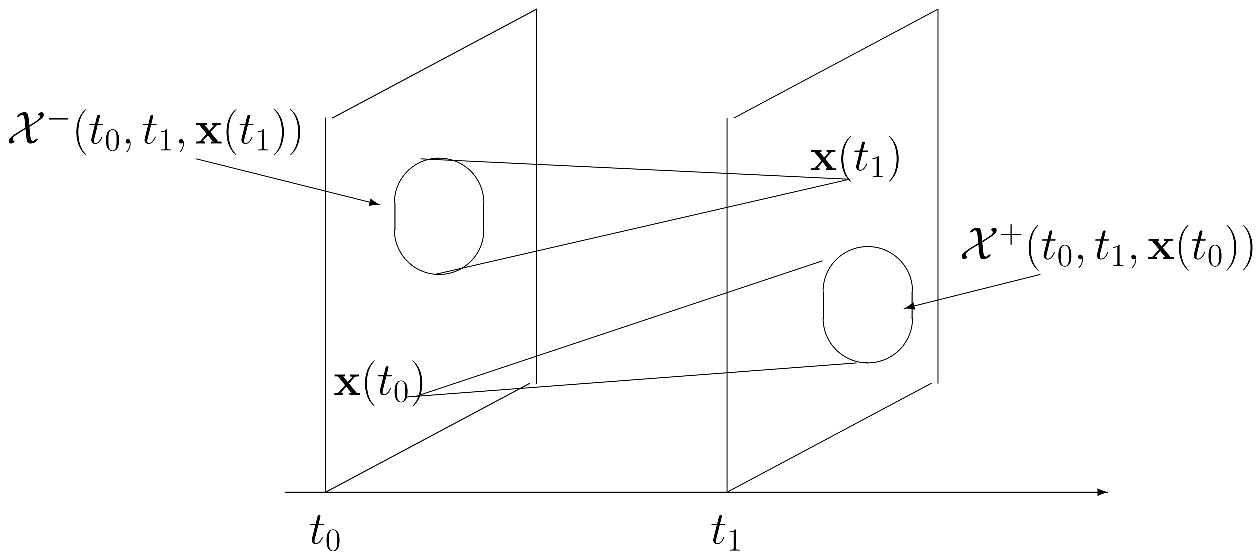
Reachability and Controllability

- **Reachability**. The reachability problem is to "find the set of all the *final states* $\mathbf{x}(t_1)$ *reachable starting from a given initial state* $\mathbf{x}(t_0)$":
  - A state $\mathbf{x}(t_1)$ of a dynamic system is reachable from the state $\mathbf{x}(t_0)$ in the time interval $[t_0, t_1]$ if it exists an input function $\mathbf{u}(\cdot) \in \mathcal{U}$ such that $\mathbf{x}(t_1) = \psi(t_0, t_1, \mathbf{x}(t_0), \mathbf{u}(\cdot))$.
  - Let $\mathcal{X}^+(t_0, t_1, \mathbf{x}(t_0))$ denote the "***set of all the final states*** $\mathbf{x}(t_1)$ ***reachable at time*** t_1 ***starting from the initial state*** $\mathbf{x}(t_0)$".
- **Controllability**. The controllability problem is "to find the set of all the *initial states* $\mathbf{x}(t_0)$ *controllable to a given* final state $\mathbf{x}(t_1)$:
  - A state $\mathbf{x}(t_0)$ of a dynamic system is controllable to state $\mathbf{x}(t_1)$ in the time interval $[t_0, t_1]$ if it exists an input function $\mathbf{u}(\cdot) \in \mathcal{U}$ such that $\mathbf{x}(t_1) = \psi(t_0, t_1, \mathbf{x}(t_0), \mathbf{u}(\cdot))$.
  - Let $\mathcal{X}^-(t_0, t_1, \mathbf{x}(t_1))$ the "***set of all the initial states*** $\mathbf{x}(t_0)$ ***controllable to the final state*** $\mathbf{x}(t_1)$ ***at time*** t_1".



- For time-invariant systems one can use $t_0 = 0$ and $t_1 = t$:

$$\mathcal{X}^+(t_0, t_1, \mathbf{x}(t_0)) \Rightarrow \mathcal{X}^+(t, \mathbf{x}(0)), \qquad \mathcal{X}^-(t_0, t_1, \mathbf{x}(t_1)) \Rightarrow \mathcal{X}^-(t, \mathbf{x}(t))$$

- For discrete-time systems it is $t \to k$:

$$\mathcal{X}^+(t, \mathbf{x}(0)) \Rightarrow \mathcal{X}^+(k, \mathbf{x}(0)), \qquad \mathcal{X}^-(t, \mathbf{x}(t)) \Rightarrow \mathcal{X}^-(k, \mathbf{x}(k))$$

Discrete time-invariant linear systems

let us consider the following discrete time-invariant linear system:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

Reachability

- The set $\mathcal{X}^+(k)$ of all the states *reachable from the origin in k* steps is equal to the set of all the states $\mathbf{x}(k)$ obtained starting from the initial condition $\mathbf{x}(0) = 0$ and considering only the forced evolution of the system:

$$\mathbf{x}(k) = \sum_{j=0}^{k-1} \mathbf{A}^{(k-j-1)}\mathbf{B}\mathbf{u}(j) = [\mathbf{B}\ \mathbf{A}\mathbf{B}\ \ldots\ \mathbf{A}^{k-1}\mathbf{B}] \begin{bmatrix} \mathbf{u}(k-1) \\ \mathbf{u}(k-2) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}$$

  and varying the input $\mathbf{u}(0), \mathbf{u}(1), \ldots, \mathbf{u}(k-1)$ in all the possible ways.

- *Definition*. *Reachability matrix in k steps*:

$$\mathcal{R}^+(k) \stackrel{\triangle}{=} [\mathbf{B}\ \mathbf{A}\mathbf{B}\ \ldots\ \mathbf{A}^{k-1}\mathbf{B}]$$

- The set $\mathcal{X}^+(k)$ of all the states *reachable from the origin in k steps* is a vectorial space which is equal to the image of matrix $\mathcal{R}^+(k)$:

$$\mathcal{X}^+(k) = \mathrm{Im}[\mathcal{R}^+(k)]$$

- The subspaces $\mathcal{X}^+(k)$ reachable in $1, 2, \ldots, k$ steps satisfy the following chain of inclusions (n is the dimension of the state space):

$$\mathcal{X}^+(1) \subseteq \mathcal{X}^+(2) \subseteq \ldots\ldots \subseteq \mathcal{X}^+(n) = \mathcal{X}^+(n+1) = \ldots$$

- The maximum reachable subspace $\mathcal{X}^+(n)$ is obtained, at the most, in n steps.

- *Definition*. *Reachability matrix* of the system:

$$\mathcal{R}^+ \stackrel{\triangle}{=} \mathcal{R}^+(k)\big|_{k=n} = \mathcal{R}^+(n) = [\mathbf{B}\ \mathbf{A}\mathbf{B}\ \ldots\ \mathbf{A}^{n-1}\mathbf{B}]$$

- The subspace $\mathcal{X}^+$ of all the state *reachable* from the origin in a time interval *however long* is equal to the image of matrix $\mathcal{R}^+$:

$$\mathcal{X}^+ = \mathrm{Im}[\mathbf{B}\ \mathbf{A}\mathbf{B}\ \ldots\ \mathbf{A}^{n-1}\mathbf{B}] = \mathrm{Im}\mathcal{R}^+$$

- *Definition*. *A system is reachable* if the subspace $\mathcal{X}^+$ of all the reachable states from the origin is equal to the whole state space $\mathbf{X}$:

$$\mathcal{X}^+ = \mathbf{X}$$

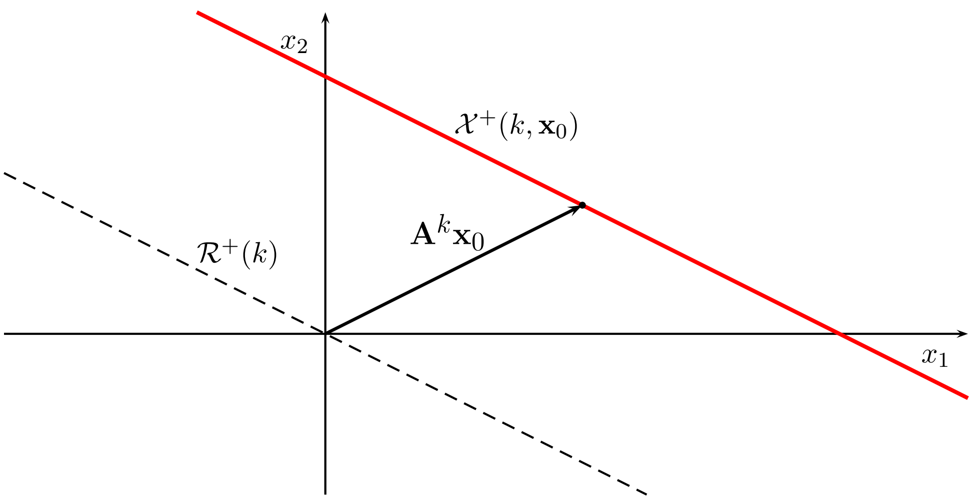
- *Necessary and sufficient* condition for a system to be reachable is:

$$\text{rank}(\mathcal{R}^+) = n$$

- For discrete, time-invariant linear systems the set $\mathcal{X}^+(k, \mathbf{x}_0)$ has the structure of a "linear variety":

$$\mathcal{X}^+(k, \mathbf{x}_0) = \mathbf{A}^k\mathbf{x}_0 + \mathrm{Im}\mathcal{R}^+(k)$$

- Graphical representation:



Controllability

- A state $\mathbf{x}_0 = \mathbf{x}(0)$ is *controllable to zero in k steps* if it exists an input sequence $\mathbf{u}(0), \mathbf{u}(1), \ldots, \mathbf{u}(k-1)$ which brings the initial state $\mathbf{x}_0$ to a final state equal to the origin $\mathbf{x}(k) = 0$ in the time interval $[0, k]$:

$$0 = \mathbf{x}(k) = \mathbf{A}^k \mathbf{x}_0 + \sum_{j=0}^{k-1} \mathbf{A}^{(k-j-1)} \mathbf{B} \mathbf{u}(j)$$

  that is:

$$-\mathbf{A}^k \mathbf{x}_0 = \sum_{j=0}^{k-1} \mathbf{A}^{(k-j-1)} \mathbf{B} \mathbf{u}(j) \quad \Rightarrow \quad \boxed{\mathbf{A}^k \mathbf{x}_0 \in \mathcal{X}^+(k) = \mathrm{Im}\mathcal{R}^+}$$

  if the state "$-\mathbf{A}^k \mathbf{x}(0)$" is reachable from the origin in k steps.

- *Property*. A system is controllable *if and only if* the following relation holds:

$$\boxed{\mathrm{Im}\mathbf{A}^n \subseteq \mathcal{X}^+(n) = \mathrm{Im}\mathcal{R}^+}$$

  where $\mathcal{R}^+$ is the reachability matrix of the system.

- For discrete linear systems the reachability and controllability properties are **NOT** equivalent:

  1) The reachability implies the controllability.

  $$\boxed{\text{reachability} \;\Rightarrow\; \text{controllability}}$$

  In fact the reachability implies $\mathcal{X}^+ = \mathbf{X}$ from which it follows that: $\mathrm{Im}\mathbf{A}^n \subseteq \mathcal{X}^+ = \mathbf{X}$, that is the system is surely controllable.

  2) The controllability does not imply the reachability:

  $$\boxed{\text{controllability} \;\not\Rightarrow\; \text{reachability}}$$

  In fact if, for example $\mathbf{A} = 0$ and $rank(\mathbf{B}) < n$, then:

  $$rank([\mathbf{B}\ \mathbf{AB}\ \ldots\ \mathbf{A}^{n-1}\mathbf{B}]) = rank([\mathbf{B}\ 0\ \ldots\ 0] < n$$

  that is the system is not reachable even if it is controllable.

- If $\mathbf{A}$ is a full rank matrix, then the reachability and the controllability imply one another.

Invariant time-continuous linear systems

Let us consider the following invariant time-continuous linear system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Reachability:

- A state $\mathbf{x}(t)$ is *reachable at time t starting from zero* if it exists an input function $\mathbf{u}(\cdot)$ such that:

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

- Let $\mathcal{X}^+(t)$ be the set of all the states reachable from the origin $\mathbf{x}(0) = 0$ in the time interval $[0,\ t]$ and let $\mathcal{X}^+$ denote be the set of all the states reachable from the origin $\mathbf{x}(0) = 0$ in the time interval $[0,\ \infty]$.

- Let R_t denote the linear function $R_t : \mathcal{U} \to \mathbf{X}$ defined as follows:

$$R_t : \mathbf{u}(\cdot) \to \mathbf{x}(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

- The set $\mathcal{U}$ is infinite dimensional. The states $\mathbf{x}(t)$ reachable at time t are all the states which belongs to the *image* of the linear function R_t:

$$\mathcal{X}^+(t) = \{\mathbf{x}\ :\ \mathbf{x} \in \mathrm{Im} R_t\}$$

- The set $\mathcal{X}^+(t)$, being the image of a linear function, *is a vectorial subspace* of the state space $\mathbf{X}$.

- *Property*. For each $t > 0$, the reachable subspace $\mathcal{X}^+(t)$ is the image of the reachability matrix $\mathcal{R}^+$:

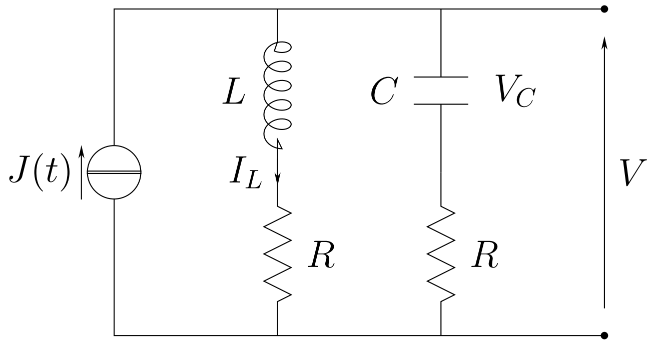
$$\boxed{\mathcal{X}^+(t) = \mathcal{X}^+ = \mathrm{Im}\mathcal{R}^+}$$

- For continuous-time systems, the reachable subspace *does NOT* depend on the length of the time interval $[0, t]$.

- The smaller is the time interval $[0, t]$ the larger is the control action $\mathbf{u}(t)$.

Controllability. For invariant time-continuous linear systems, the controllable subspace $\mathcal{X}^-$ does not depend on the amplitude of the time interval $[0, t]$ and it **is equal** to the reachable subspace $\mathcal{X}^+$.

Example. Let us consider the following electrical network:



The dynamic equations of the systems are:

$$\begin{cases} L\dfrac{d\,I_L}{dt} &= V_C + R(J - I_L) - R\,I_L \\ C\dfrac{d\,V_C}{dt} &= J - I_L \\ V &= V_C + R(J - I_L) \end{cases}$$

where I_L is the current which flows in the inductance, V_C is the voltage across the capacitor, J is the input current and V is the output voltage. In matrix form, the system dynamics can be represented as follows:

$$\begin{cases} \dot{\mathbf{x}} &= \begin{bmatrix} \frac{-2R}{L} & \frac{1}{L} \\ \frac{-1}{C} & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{R}{L} \\ \frac{1}{C} \end{bmatrix} J \\ V &= \begin{bmatrix} -R & 1 \end{bmatrix}\mathbf{x} + [R]J \end{cases} \qquad \mathbf{x} = \begin{bmatrix} I_L \\ V_C \end{bmatrix}$$

The reachability matrix of the system is

$$\mathcal{R}^+ = \begin{bmatrix} \frac{R}{L} & \frac{1}{LC} - \frac{2R^2}{L^2} \\ \frac{1}{C} & -\frac{R}{LC} \end{bmatrix}, \qquad \det \mathcal{R}^+ = \frac{1}{LC}\left[\frac{R^2}{L} - \frac{1}{C}\right]$$

The system is reachable only if $\mathcal{R}^+$ is a full rank matrix. The system is NOT completely reachable if:

$$R^2 = \frac{L}{C} \qquad \leftrightarrow \qquad R\,C = \frac{L}{R}$$

that is if the inductance time constant $\frac{L}{R}$ is equal to the capacitor time constant RC. In this case the two system eigenvalues are coincident: $\lambda_{1,2} = -\frac{1}{\sqrt{LC}}$.

Example. Compute the reachability matrix $\mathcal{R}^+$ of the following system:

$$\left\{ \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u(t) \right.$$

Reachability matrix $\mathcal{R}^+$ and computation of the subspace $\mathcal{X}^+$:

$$\mathcal{R}^+ = \left[\begin{array}{ccc} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} \end{array} \right] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \qquad \mathcal{X}^+ = \mathrm{Im}\mathcal{R}^+ = \mathrm{Im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system is reachable.

Example. Compute the reachability matrix $\mathcal{R}^+$ of the following system:

$$\left\{ \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t) \right.$$

Reachability matrix $\mathcal{R}^+$ and computation of the subspace $\mathcal{X}^+$:

$$\mathcal{R}^+ = \left[\begin{array}{ccc} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} \end{array} \right] = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{X}^+ = \mathrm{Im}\left[\mathcal{R}^+\right] = \mathrm{Im} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The system is NOT completely reachable.

Example. Compute the reachability matrix $\mathcal{R}^+$ of the following system:

$$\left\{ \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u(t) \right.$$

Reachability matrix $\mathcal{R}^+$ and computation of the subspace $\mathcal{X}^+$:

$$\mathcal{R}^+ = \left[\begin{array}{ccc} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{array} \right] = \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right], \qquad \mathcal{X}^+ = \mathrm{Im}\left[\mathcal{R}^+\right] = \mathrm{Im} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The system is NOT completely reachable

Equivalent systems

- *Property*. Discrete or continuous-time linear systems which are *algebraically equivalent* have the same reachability properties.
- Let $\mathbf{T}$ be a full rank transformation matrix which links two algebraically equivalent time-invariant linear systems $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$ and $\bar{\mathcal{S}} = (\overline{\mathbf{A}},\ \overline{\mathbf{B}},\ \overline{\mathbf{C}},\ \overline{\mathbf{D}})$:

$$\mathbf{x} = \mathbf{T}\overline{\mathbf{x}} \qquad \rightarrow \qquad \begin{cases} \overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, & \overline{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B} \\ \overline{\mathbf{C}} = \mathbf{C}\mathbf{T}, & \overline{\mathbf{D}} = \mathbf{D} \end{cases}$$

The reachability subspaces in k steps of the two systems $\mathcal{S}$ and $\bar{\mathcal{S}}$, that is $\mathcal{X}^+(k)$ and $\overline{\mathcal{X}}^+(k)$, are linked by the following relation:

$$\overline{\mathcal{X}}^+(k) = \mathrm{Im}[\overline{\mathbf{B}} \ldots \overline{\mathbf{A}}^{k-1}\overline{\mathbf{B}}] = \mathrm{Im}(\mathbf{T}^{-1}[\mathbf{B} \ldots \mathbf{A}^{k-1}\mathbf{B}]) = \mathbf{T}^{-1}\mathcal{X}^+(k)$$

- The subspace $\mathcal{X}^+$ is invariant with respect to a state space transformation:

$$\mathbf{x} = \mathbf{T}\overline{\mathbf{x}} \qquad \rightarrow \qquad \boxed{\mathcal{X}^+ = \mathbf{T}\overline{\mathcal{X}}^+}$$

- Let $\mathcal{R}^+$ and $\overline{\mathcal{R}}^+$ be the *reachability matrices* of the two systems. The following relation holds:

$$\boxed{\overline{\mathcal{R}}^+ = \mathbf{T}^{-1}\mathcal{R}^+} \qquad \Leftrightarrow \qquad \boxed{\mathcal{R}^+ = \mathbf{T}\,\overline{\mathcal{R}}^+}$$

- If the two systems have only one input, $\mathcal{R}^+$ and $\overline{\mathcal{R}}^+$ are squared full rank matrices which satisfy the following relation:

$$\boxed{\mathbf{T} = \mathcal{R}^+(\overline{\mathcal{R}}^+)^{-1}}$$