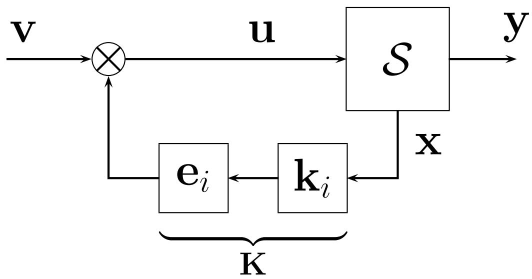
Pole placement: the general case

- *Property. Let $\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D})$ be a linear system of dimension n, with m inputs and completely reachable.*

  For each given minic polynomial $p(\lambda)$ of degree n, it exists a matrix $\mathbf{K} \in \mathcal{R}^{m\times n}$ such that the characteristic polynomial $\Delta_{\mathbf{A+BK}}$ of matrix $\mathbf{A}+\mathbf{BK}$ of the feedback system $\mathcal{S}_{\mathbf{K}}$ is equal to polynomial $p(\lambda)$.

- Proof. The following two cases are considered:

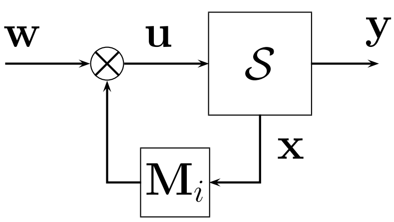
  1) *The system is reachable using only the i-th input.* In this case it is possible to palace arbitrarily the eigenvalues of the feedback system using only the i-th input:



$$\mathbf{K} = \mathbf{e}_i\mathbf{k}_i = \begin{bmatrix} 0 \\ \vdots \\ \mathbf{k}_i \\ \vdots \\ 0 \end{bmatrix}$$

  In this case the feedback gain matrix $\mathbf{K}$ has only the i-th row which is nonzero.

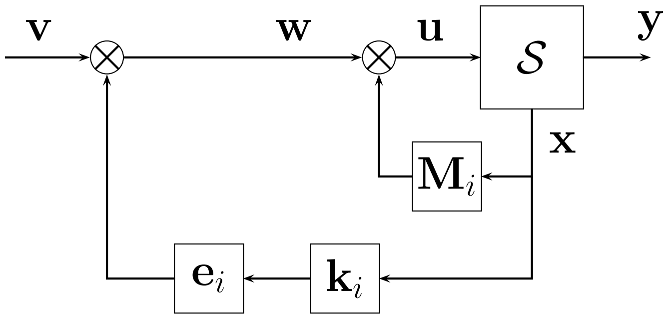
  2) *The system is reachable only using two or more inputs.* In this case a first feedback control law $\mathbf{u} = \mathbf{M}_i\mathbf{x} + \mathbf{w}$ is designed such that the feedback system is reachable using only the i-th input:

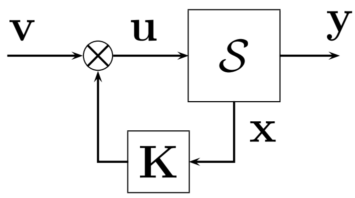


$$\begin{cases} \dot{\mathbf{x}} &= (\mathbf{A}+\mathbf{BM}_i)\mathbf{x} + \mathbf{Bw} \\ \mathbf{y} &= (\mathbf{C}+\mathbf{DM}_i)\mathbf{x} + \mathbf{Dw} \end{cases}$$

  Matrix $\mathbf{M}_i$ can be computed using the Heyman lemma.

Then a second static feedback control law $\mathbf{w} = \mathbf{e}_i\mathbf{k}_i\mathbf{x} + \mathbf{v}$ is designed in order to locate arbitrarily the eigenvalues of the feedback system:





The feedback gain matrix $\mathbf{K}$ has the following structure:

$$\mathbf{K} = \mathbf{M}_i + \mathbf{e}_i\mathbf{k}_i, \qquad \begin{cases} \dot{\mathbf{x}} = [\mathbf{A} + \mathbf{B}(\mathbf{M}_i + \mathbf{e}_i\mathbf{k}_i)]\mathbf{x} + \mathbf{B}\mathbf{v} \\ \mathbf{y} = [\mathbf{C} + \mathbf{D}(\mathbf{M}_i + \mathbf{e}_i\mathbf{k}_i)]\mathbf{x} + \mathbf{D}\mathbf{v} \end{cases}$$

- So, if the couple $(\mathbf{A},\ \mathbf{B})$ is reachable, gli eigenvalues of matrix $\mathbf{A} + \mathbf{BK}$ can be chosen arbitrarily and therefore the feedback system can be stabilized as much as required.

- on the contrary, if the couple $(\mathbf{A},\ \mathbf{B})$ is not reachable, only the eigenvalues of the reachable part of the system can be arbitrarily located. In this case the feedback control law can be used to stabilize the system *only if the not reachable subsystem is asymptotically stable*.

- Heymann lemma. *If system $(\mathbf{A}, \mathbf{B})$ reachable and if $\mathbf{b}_i$ is a nonzero column of matrix $\mathbf{B}$, then it exists a matrix $\mathbf{M}_i \in \mathcal{R}^{m\times n}$, such that system $(\mathbf{A} + \mathbf{B}\mathbf{M}_i\ ,\ \mathbf{b}_i)$ is reachable.*

- Computation of matrix $\mathbf{M}_i$ when $i = 1$: chose n linearly independent columns of the reachability matrix $\mathcal{R}^+$ as follows:

  a) Chose the vectors $\mathbf{A}^i\mathbf{b}_1$, for $i = 1, \ldots, \nu_1$, up to the value ν_1 such that $\mathbf{A}^{\nu_1}\mathbf{b}_1$ is linearly dependent on the preceding vectors:

  $$\mathbf{A}^{\nu_1}\mathbf{b}_1 \in \mathrm{Im}\{\mathbf{b}_1,\ \mathbf{A}\mathbf{b}_1,\ , \ldots, \mathbf{A}^{\nu_1-1}\mathbf{b}_1\}$$

  b) Similarly, chose the vectors $\mathbf{A}^i\mathbf{b}_2$, for $i = 1, \ldots, \nu_2$, such that:

  $$\mathbf{A}^{\nu_2}\mathbf{b}_2 \in \mathrm{Im}\{\mathbf{b}_1,\ , \ldots, \mathbf{A}^{\nu_1-1}\mathbf{b}_1,\ \mathbf{b}_2,\ , \ldots, \mathbf{A}^{\nu_2-1}\mathbf{b}_2\}$$

  The procedure stops when $\nu_1 + \nu_2 + \ldots + \nu_k = n$.

c) Let matrices $\mathbf{Q} \in \mathcal{R}^{n\times n}$ and $\mathbf{S} \in \mathcal{R}^{m\times n}$ be defined as follows:

$$\mathbf{Q} = [\mathbf{b}_1 \;\ldots\; \mathbf{A}^{\nu_1-1}\mathbf{b}_1 \,|\, \ldots \,|\, \mathbf{b}_{k-1} \;\ldots\; \mathbf{A}^{\nu_{k-1}-1}\mathbf{b}_{k-1} \,|\, \mathbf{b}_k \;\ldots\; \mathbf{A}^{\nu_k-1}\mathbf{b}_k]$$

$$\mathbf{S} = [0 \;\ldots\; \mathbf{e}_2 \,|\, \ldots \,|\, 0 \;\ldots\; \mathbf{e}_k \,|\, 0 \;\ldots\; 0]$$

where vector $\mathbf{e}_i$ denotes the i-th column of the identity matrix $\mathbf{I}_m$. Vectors $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, etc. are located in correspondence of the ν_1-th, $(\nu_1+\nu_2)$-th, $(\nu_1+\nu_2+\nu_3)$-th, etc. column of matrix $\mathbf{S}$.

d) The matrix

$$\mathbf{M}_1 = \mathbf{S}\mathbf{Q}^{-1}$$

satisfies the Heymann lemma.

Example. let us consider the following linear system ($\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$) with two inputs:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t), \qquad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$$

Design, if it's possible, a static feed control law $\mathbf{u} = \mathbf{K}\mathbf{x}$ which locates in -1 all the eigenvalues of the feedback system.

<u>Solution</u>. The reachability subspaces obtained using the first and the second input are:

$$\mathcal{X}^+_{\mathbf{b}1} = \mathrm{Im} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{Im} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{X}^+_{\mathbf{b}2} = \mathrm{Im} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \mathrm{Im} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

The system is not reachable using only one input. All the eigenvalues of matrix $\mathbf{A}$ are unstable, $\lambda_{1,2,3} = 1$, and therefore if only one input is used the not reachable part of the system is surely unstable: the system cannot be stabilized using only one input. The system is completely reachable if both the inputs are used, and therefore surely it exists a gain matrix $\mathbf{K}$ such that the eigenvalues of the feedback system can be chosen arbitrarily. Let us use the Heymann lemma to make the system reachable using, for example, the first input. The matrices $\mathbf{Q}$, $\mathbf{S}$ and $\mathbf{M}_1$ have the following structure

$$\mathbf{Q} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{A}\mathbf{b}_1 \,|\, \mathbf{b}_2 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad \rightarrow \qquad \mathbf{Q}^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{S} = \left[\begin{array}{cc|c} 0 & \mathbf{e}_2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\mathbf{M}_1 = \mathbf{S}\mathbf{Q}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using matrix $\mathbf{M}_1$ one obtains the following intermediate system

$$\mathbf{A} + \mathbf{B}\mathbf{M}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The characteristic polynomial of this matrix is

$$\Delta_{\mathbf{A}+\mathbf{B}\mathbf{M}_1}(s) = s^3 - 3s^2 + 2s = s(s-1)(s-2)$$

The desired characteristic polynomial is

$$\Delta_{\mathbf{A}+\mathbf{B}\mathbf{K}}(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$$

The reachability matrix of the sistema $(\mathbf{A} + \mathbf{B}\mathbf{M}_1,\ \mathbf{b}_1)$ is:

$$\mathcal{R}^+ = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The gain matrix $\mathbf{k}_1^T$ which places in -1 all the eigenvalues of the feedback system is:

$$\begin{aligned}
\mathbf{k}_1^T &= \mathbf{k}_c^T[\mathcal{R}^+(\mathcal{R}_c^+)^{-1}]^{-1} \\
&= \begin{bmatrix} -1 & -1 & -6 \end{bmatrix} \left\{ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}^{-1} \\
&= \begin{bmatrix} -1 & -1 & -6 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -1 & -1 & -6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -13 & -6 & -14 \end{bmatrix}
\end{aligned}$$

So, the following global gain matrix $\mathbf{K}$ has been obtained:

$$\mathbf{K} = \mathbf{M}_1 + \begin{bmatrix} \mathbf{k}_1^T \\ 0 \end{bmatrix} = \begin{bmatrix} -13 & -6 & -14 \\ 1 & 0 & 0 \end{bmatrix}$$

When two or more inputs are used, the pole placement problems has infinite solutions.

In the considered case, for example, a second solution of the pole placement problem which does use the Heymann lemma is the following. Let us consider the generic matrix $\mathbf{K}$ with six unknown parameters:

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix}$$

The matrix $\mathbf{A}+\mathbf{BK}$ of the feedback system has the following structure:

$$\mathbf{A}+\mathbf{BK} = \begin{bmatrix} 1 & 1 & 1 \\ k_{11} & 1+k_{12} & k_{13} \\ k_{21} & k_{22} & 1+k_{23} \end{bmatrix}$$

Choosing $k_{21} = 0$, $k_{22} = 0$ and $k_{13} = 0$ one obtains the following block diagonal matrix

$$\mathbf{A}+\mathbf{BK} = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ k_{11} & 1+k_{12} & 0 \\ \hline 0 & 0 & 1+k_{23} \end{array}\right]$$

All the eigenvalues of this feedback matrix can be located -1 if, for example, we choose $k_{23} = -2$ and we impose

$$\Delta_{\mathbf{A}+\mathbf{BK}}(s) = s^2 - (2+k_{12})s + 1 + k_{12} - k_{11} = s^2 + 2s + 1$$

From these relations one obtains

$$k_{12} = -4, \qquad\qquad k_{11} = -4$$

So, finally, the gain matrix has the following structure:

$$\mathbf{K} = \begin{bmatrix} -4 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$