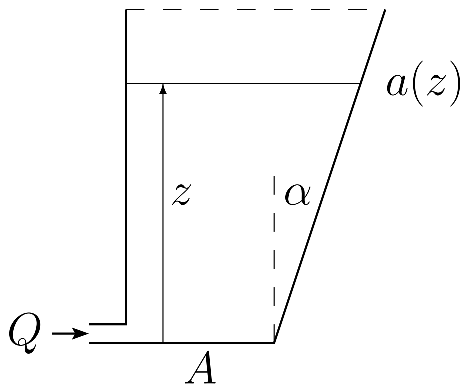
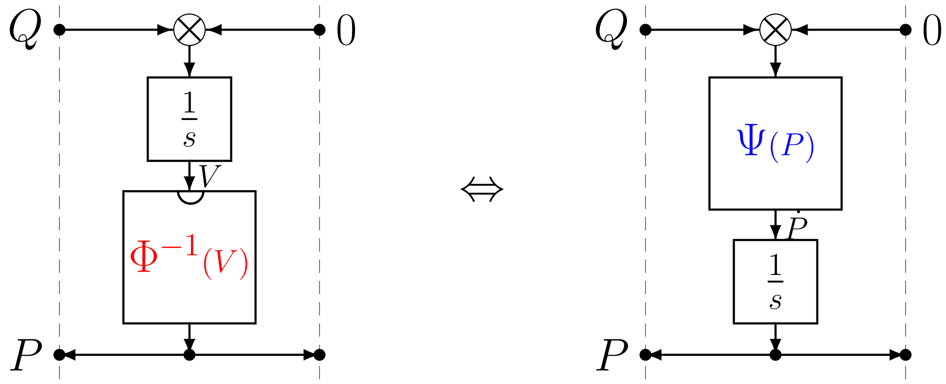
- Example. Tank with variable section.

Q	input volume flow rate
P	input pressure
V	volume of liquid in the tank
z	height of the liquid
$a(z)$	area of the liquid at height z
A	area of the base section
α	slope of the liquid area
p_0	pressure per unit of height



- Two equivalent POG block schemes can be used:



- The area of the liquid $a(z)$ at height z and the volume of liquid V are:

$$a(z) = A + \alpha z, \qquad V = \int_0^z a(z)dz = A\,z + \frac{\alpha\,z^2}{2}$$

- Since $P = p_0\,z$, the constitutive relation $V = \Phi(P)$ is defined as follows:

$$z = \frac{P}{p_0}, \qquad V = \Phi(P) = A\,\frac{P}{p_0} + \frac{\alpha\,P^2}{2p_0^2}.$$

- The inverse function $P = \Phi^{-1}(V)$ is defined as follows:

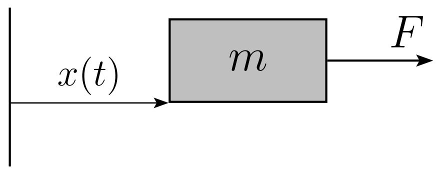
$$P = \Phi^{-1}(V) = p_0\frac{\sqrt{A^2 + 2\alpha\,V} - A}{\alpha}$$

- It can be proved that function $\Psi(P)$ can be obtained as follows:

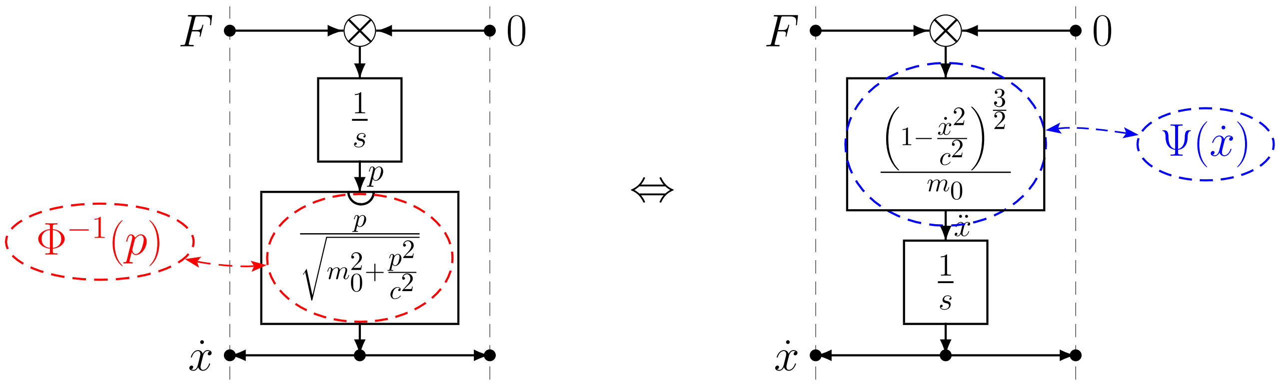
$$\Psi(P) = \left[\frac{\partial\Phi(P)}{\partial P}\right]^{-1} = \left[\frac{A}{p_0} + \frac{\alpha\,P}{p_0^2}\right]^{-1} = \frac{p_0^2}{Ap_0 + \alpha P}$$

- Example. A translating mass.

x	position of the mass
$\dot{x}$	velocity of the mass
m	value of the mass
m_0	value of the mass at rest
c	velocity of the light
p	momentum of the mass



- The system can be described using one of the following POG schemes:



- Taking into account the relativistic effects, mass m and $p = \Phi(\dot{x})$ are:

$$m = \frac{m_0}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}, \qquad p = \Phi(\dot{x}) = m\,\dot{x} = \frac{m_0\,\dot{x}}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}$$

- The inverse function $\Phi^{-1}(p)$ can be expressed as follows:

$$\dot{x} = \Phi^{-1}(p) = \frac{p}{\sqrt{m_0^2 + \frac{p^2}{c^2}}}$$

- Function $\Psi(\dot{x})$ can be computed as follows:

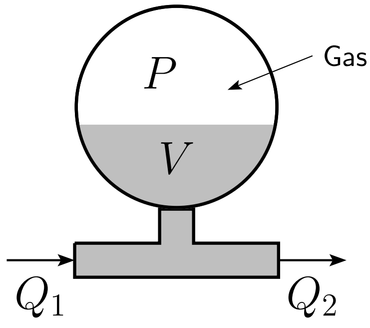
$$\Psi(\dot{x}) = \left[\frac{\partial \Phi(\dot{x})}{\partial \dot{x}}\right]^{-1} = \left[\frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}\right]^{-1} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}{m_0}$$

- The energy E_s stored in the system can be expressed as follows:

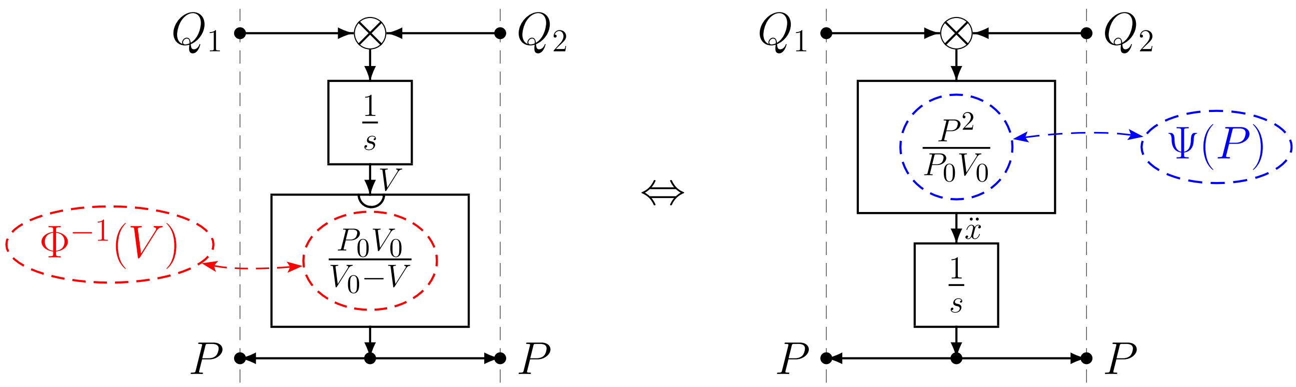
$$E_s = \int_0^p \Phi^{-1}(p)\,dp = \int_0^p \frac{p}{\sqrt{m_0^2 + \frac{p^2}{c^2}}}\,dp = m_0\,c^2 \left[\sqrt{1 + \frac{p^2}{m_0^2 c^2}} - 1\right]$$

- Nonlinear hydraulic capacitor.

Q_1	input volume flow rate
Q_2	output volume flow rate
P	pressure of the capacitor
V	volume of liquid within the capacitor
V_G	volume of gas within the capacitor
P_0	pressure of the empty capacitor
V_0	volume of the empty capacitor



- The system can be described using the following POG schemes:



- The gas pressure P within the capacitor satisfies the following equation:

$$P\,V_G = P_0V_0, \qquad V_G = V_0 - V \qquad \rightarrow \qquad P = \Phi^{-1}(V) = \frac{P_0V_0}{V_0 - V}$$

- The constitutive equation of the nonlinear hydraulic capacitor is:

$$V = \Phi(P) = V_0 - \frac{P_0V_0}{P}$$

- Function $\Psi(P)$ can be computed as follows:

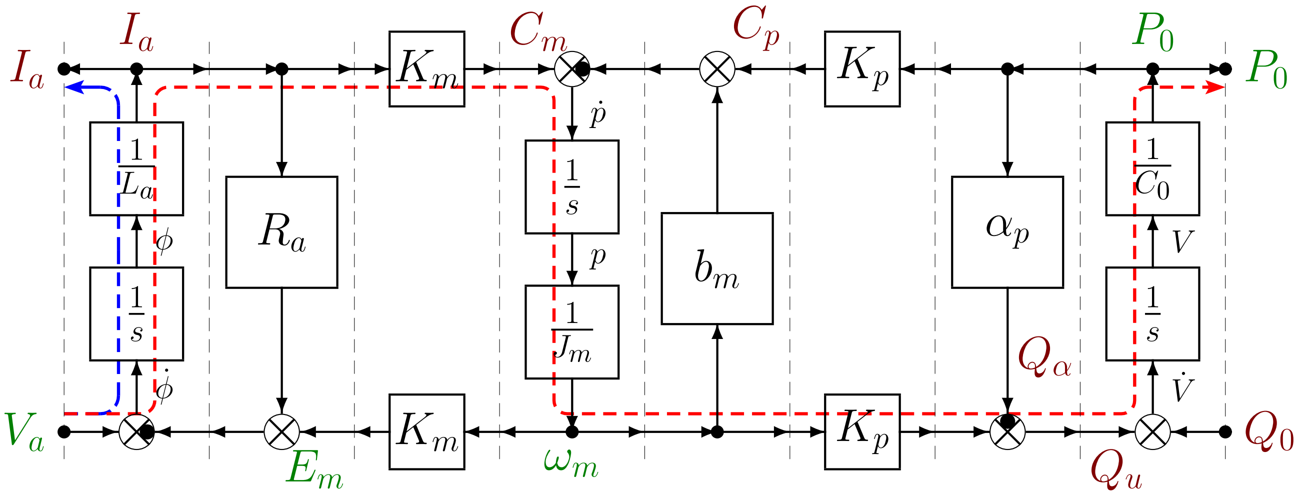
$$\Psi(P) = \left[\frac{\partial\Phi(P)}{\partial P}\right]^{-1} = \left[\frac{P_0V_0}{P^2}\right]^{-1} = \frac{P^2}{P_0V_0}$$

- The energy E_s stored in the system can be expressed as follows:

$$E_s = \int_0^V \Phi^{-1}(p)\,dp = \int_0^V \frac{P_0V_0}{V_0 - V}\,dV = P_0V_0 \ln\frac{V_0}{(V_0 - V)}$$

Relative degree of a transfer function $G(s)$

- Let us consider a generic block scheme:



- For each transfer function $G(s) = \frac{Y(s)}{U(s)}$ which links an input $u(t)$ to the output $y(t)$, the following properties hold:

  1) the order of function $G(s)$ is equal to the number n of independent dynamic elements which store energy within the system;
  2) the poles of function $G(s)$ are equal to the solutions of equation $\Delta(s) = 0$ where $\Delta(s)$ is the determinant of the block scheme;
  3) the **relative degree** of function $G(s)$ is equal to the minimum number r of integrators which is present in the set of all the paths that links the input $u(t)$ to the output $y(t)$;
  4) if there is only one path $\mathcal{P}_1$ that links the input $u(t)$ to the output $y(t)$, then the zeros of function $G(s)$ are equal to the solutions of equation $\Delta_1(s) = 0$ where $\Delta_1(s)$ is the determinant of the reduced block scheme obtained from the original one eliminating all the blocks touched by the path $\mathcal{P}_1$;

  $G(s) = \frac{P_0}{V_a}$ has 3 poles and 0 zeros because the relative degree is $r = 3$;

  $G(s) = \frac{I_a}{V_a}$ has 3 poles and 2 zeros because the relative degree is $r = 1$;

- Note: the higher is the relative degree the more difficult is the control.