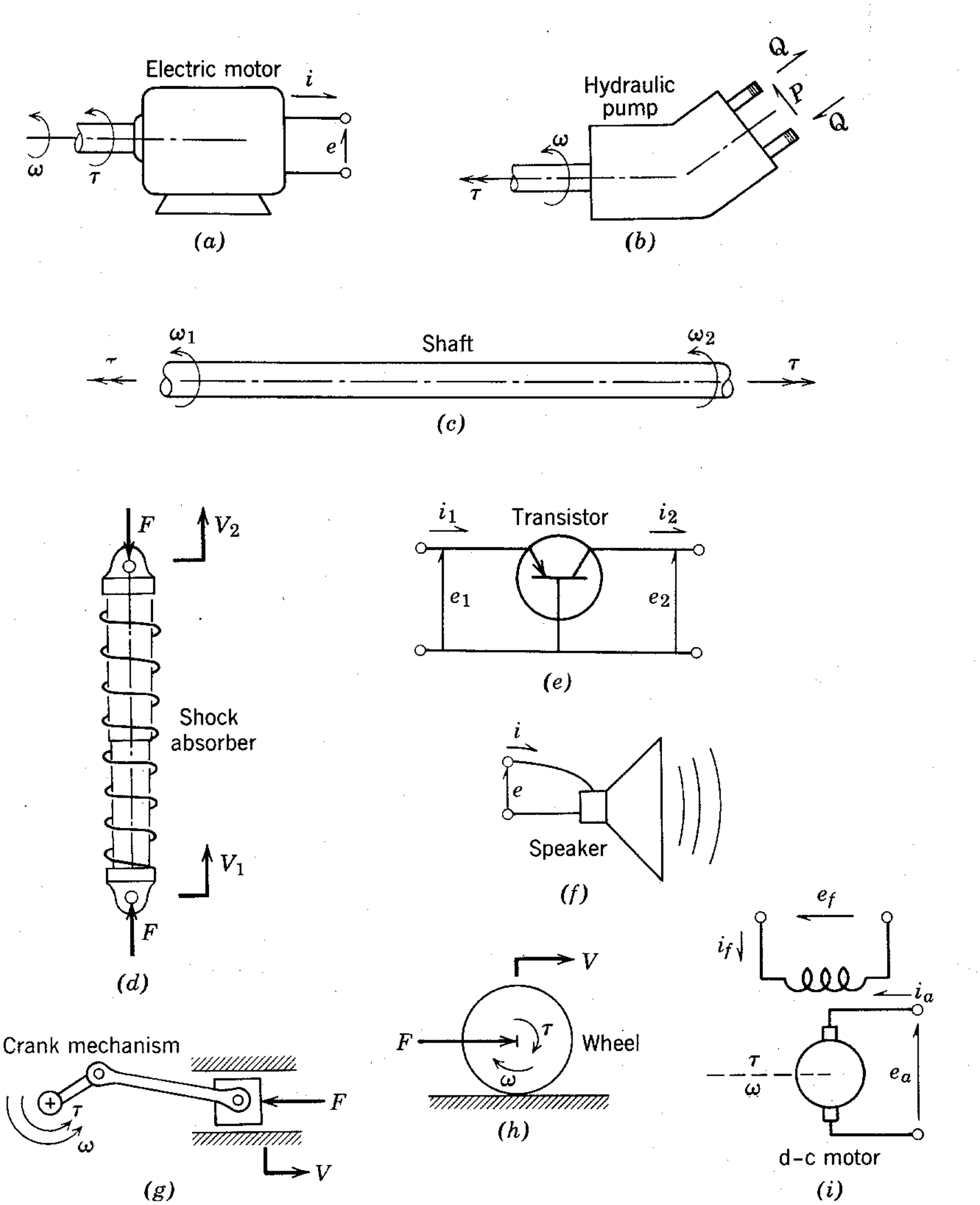
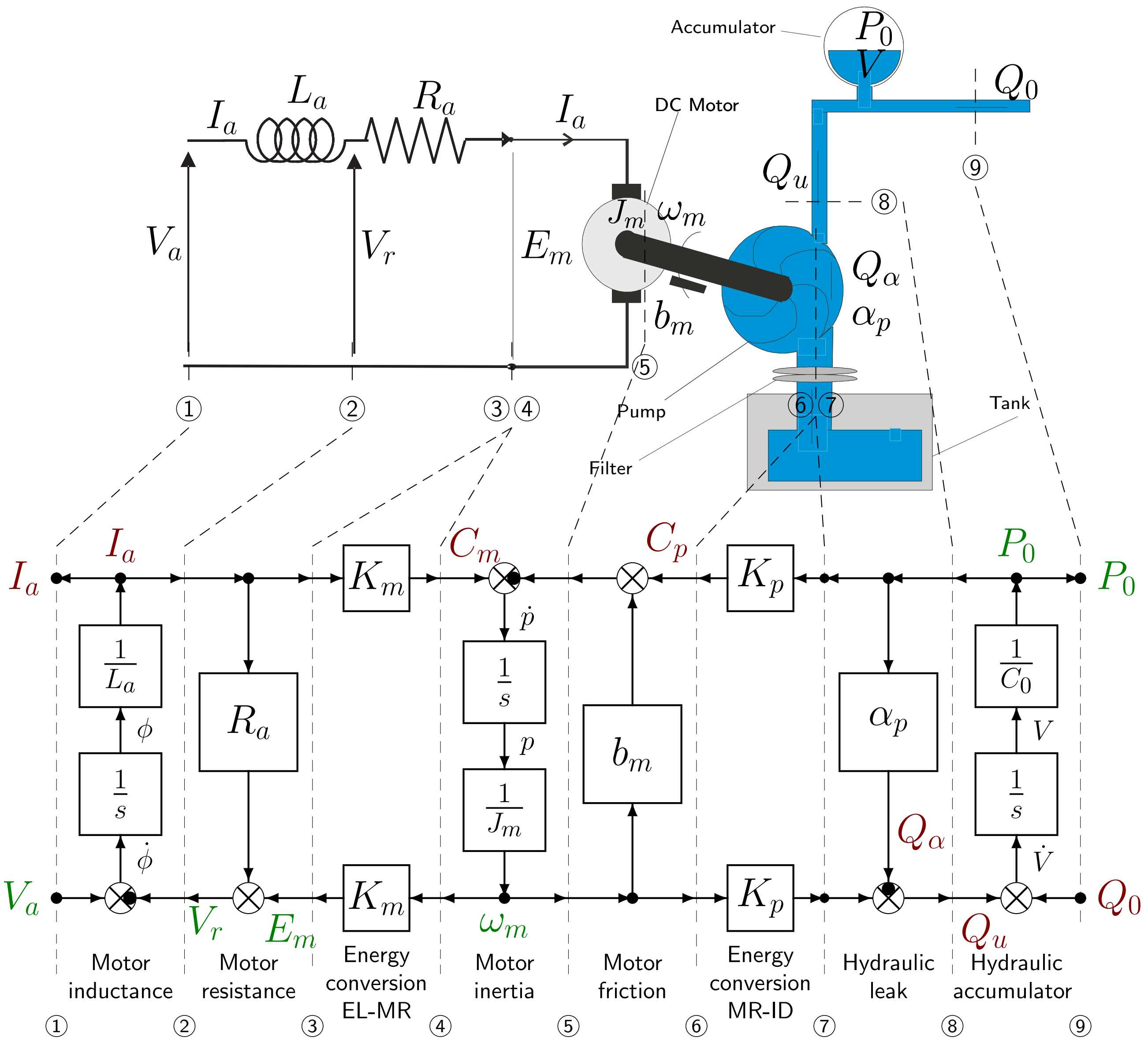
The Power-Oriented Graphs Modeling Technique

- Complex physical systems can always be decomposed in basic physical elements which interact with each other by means of "energetic ports", and "power flows".
- Examples of elementary physical systems:



- **The Power-Oriented Graphs** (POG):
  - is a graphical modeling techniques that uses an "*energetic approach*" for modeling physical systems.
  - use the "*power*" and "*energy*" variables as basic concepts for modeling physical systems.
  - the POG block schemes are easy to use, easy to understand and can be directly implemented in Simulink.
  - is based on the same energetic concept of the Bond Graph modeling technique. See: Karnopp, Margolis, Rosemberg, "*System Dynamics - A unified approach*", John Wiley & Sons.
- ***Example***. A DC electric motor moves an hydraulic pump. The physical system and the corresponding POG block scheme:

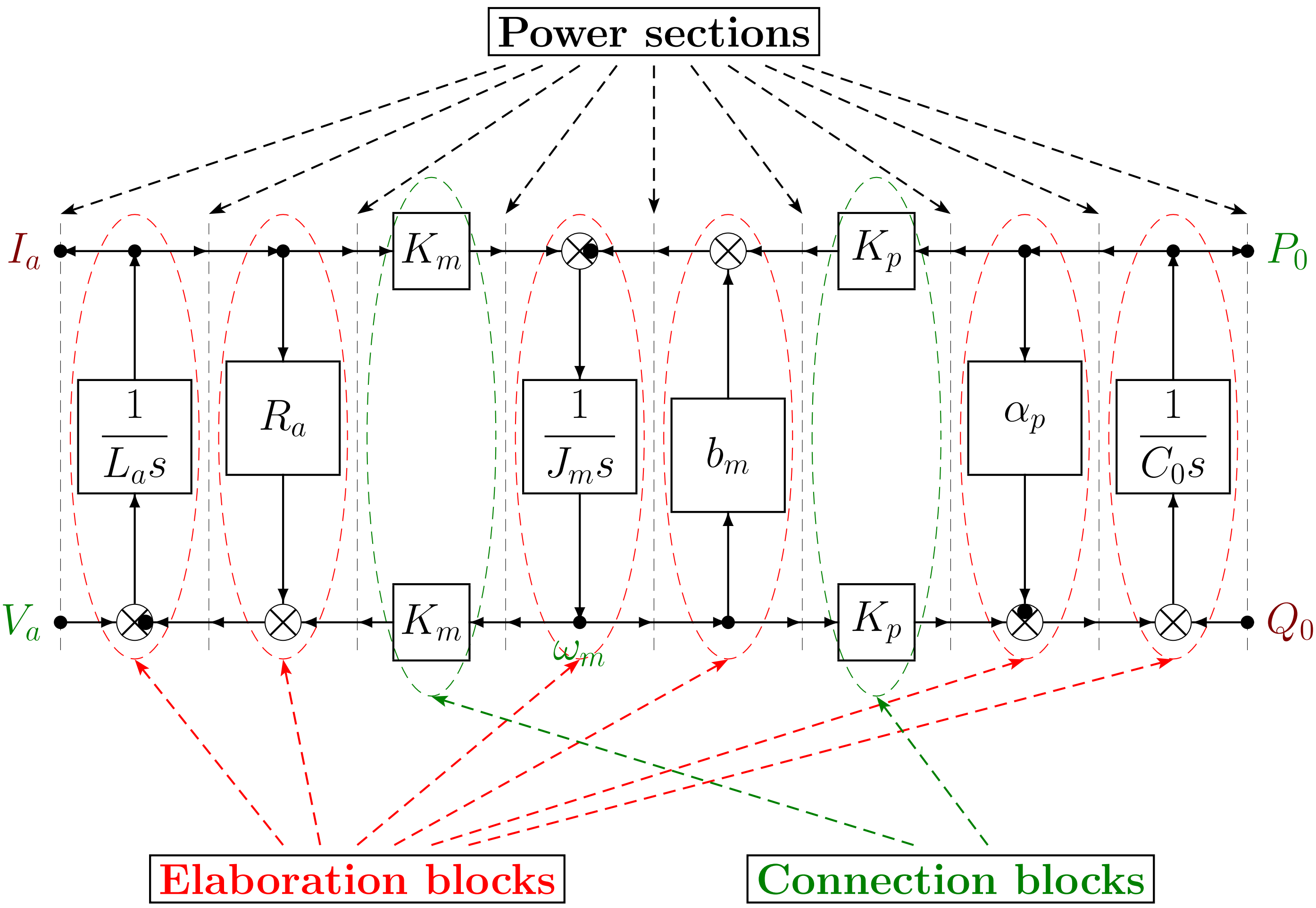


- The "*energetic approach*" is useful for modeling because the physical systems are always characterized by the following properties:

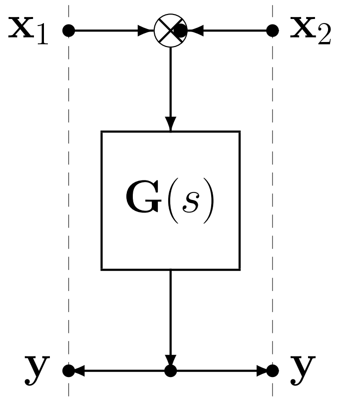
  1) a physical system "*stores and/or dissipates energy*";
  2) the dynamic model of a physical system describes "*how the energy moves*" within the system,
  3) the energy moves from point to point within the system only by means of two "*power variables*".

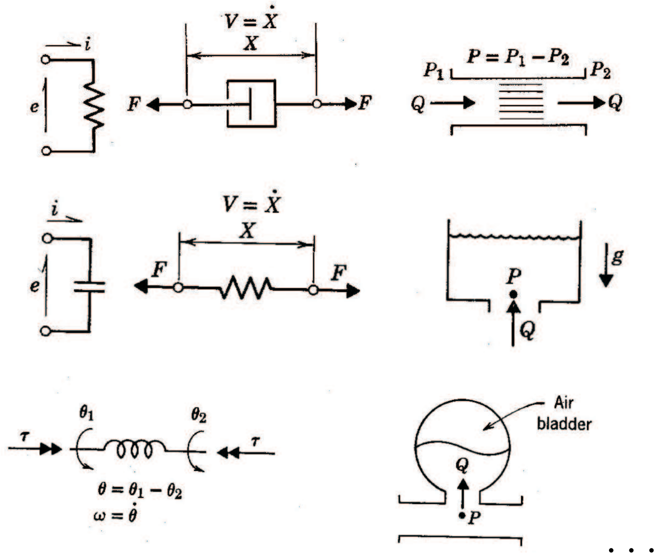
- ***Power sections.*** The "dashed lines" of the POG schemes represent the "power sections" of the system. The inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ of the two "power variables" $\mathbf{x}$ and $\mathbf{y}$ touched by the dashed line has the physical meaning of "*power flowing through the section*".

- ***POG blocks.*** The POG technique uses **two blocks** for modeling physical systems: the ***Elaboration block*** and the ***Connection block***.

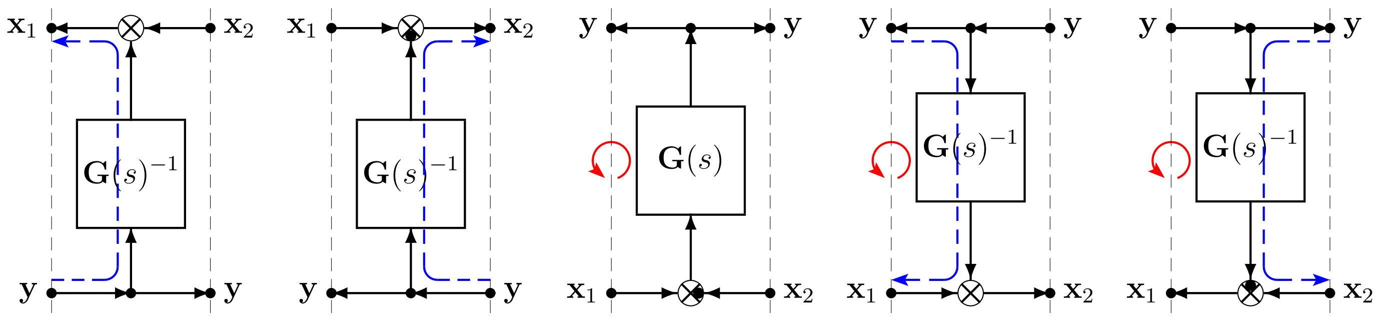


- The ***Elaboration block*** is used for modeling the physical elements that store and/or dissipate energy (i.e. springs, masses, dampers, capacities, inductances, resistances, etc.).





- Equivalent ways of representing the elaboration block:



Input path inversion | Output path inversion | Upside down | Input path inversion + upside down | Output path inversion + upside down

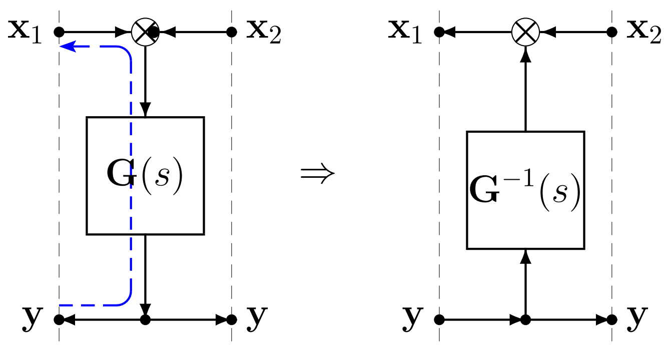
- The **black spot** within the summation element represents, when it is present, a **minus sign** that multiplies the entering variable.

- **Rules for inverting a path**:

  1) Invert each line of the path;

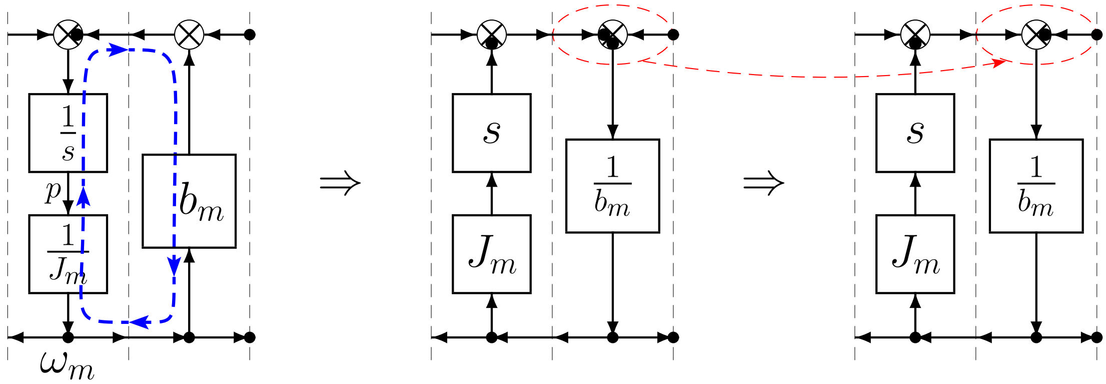
  2) Invert each block of the path;

  3) In the summation blocks invert the sign of the variables which belong to the path;



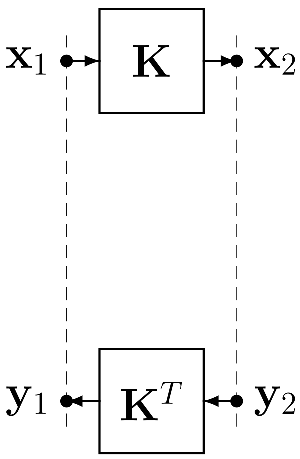
- **Example of path inversion**:

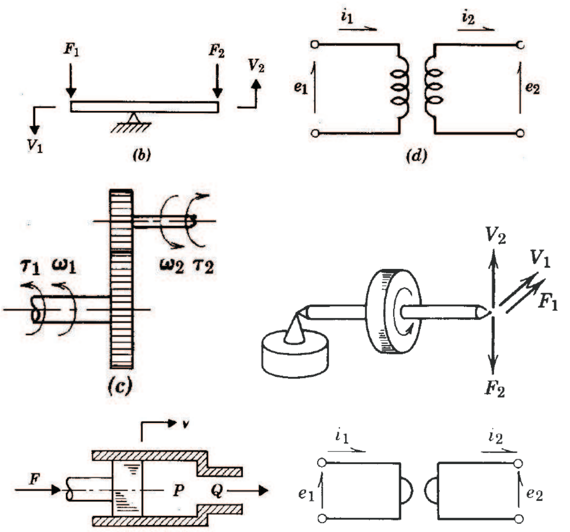
a) Path to be inverted b) Inverted path c) Equivalent scheme



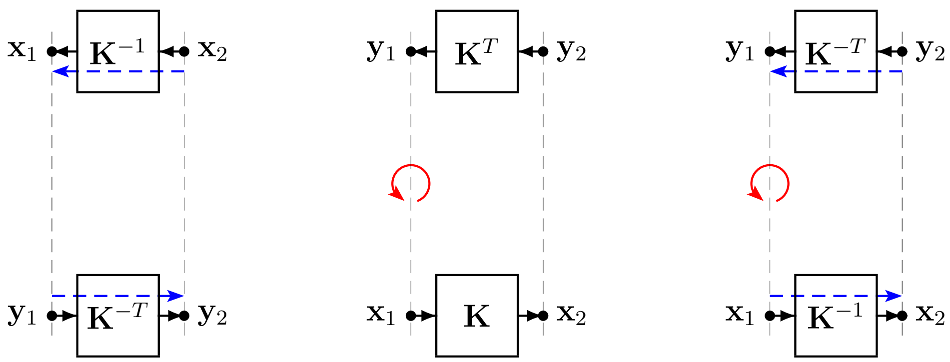
The POG block scheme does not change is the signs of a summation block are all switched to the opposite value.

- The ***Connection block*** is used for modeling the physical elements that "transform the power without losses" (i.e. *neutral elements* such as gear reductions, transformers, etc.).





- Equivalent ways of representing the connection block:



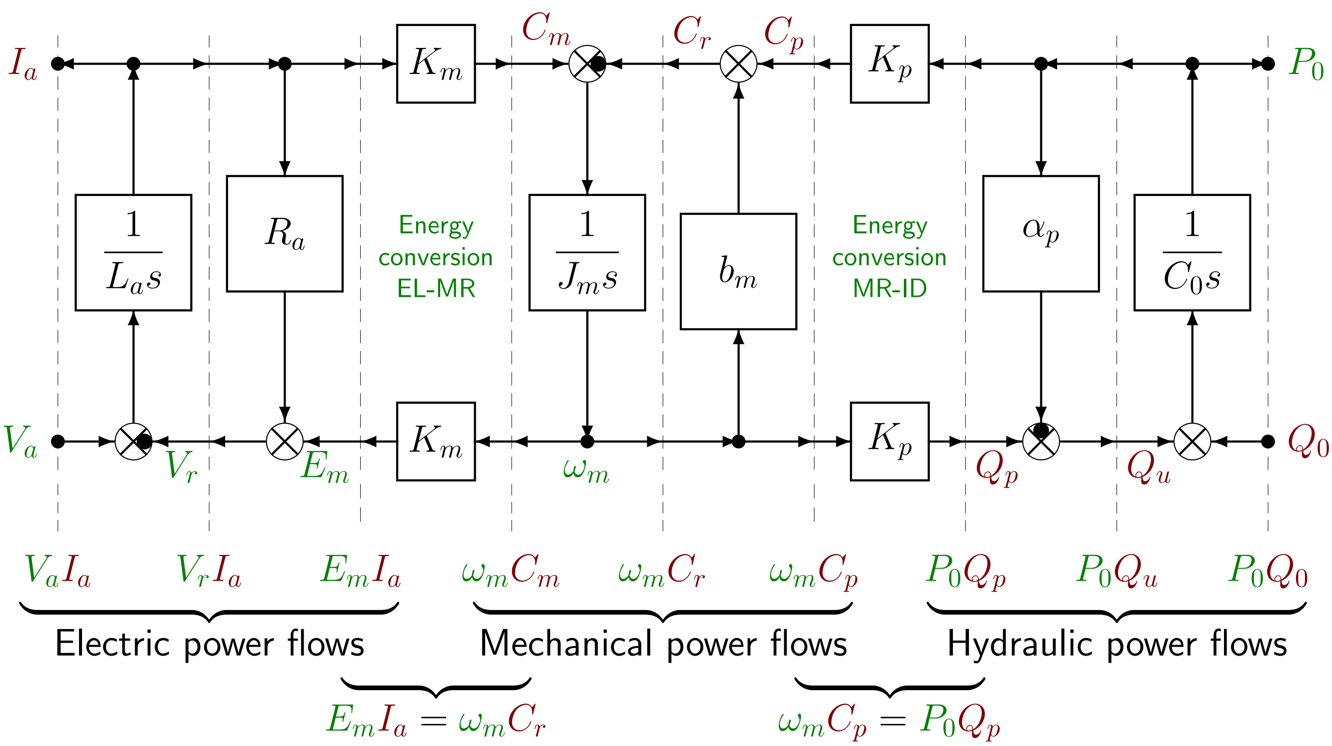
Inverted path Upside down Inverted path + upside down

- Matrix $\mathbf{K}$ can also be rectangular or time varying.

- The main ***Energetic domains*** encountered in modeling physical systems are: electrical, mechanical (translational and rotational) and hydraulic. Each energetic domain is characterized by two ***power variables***.

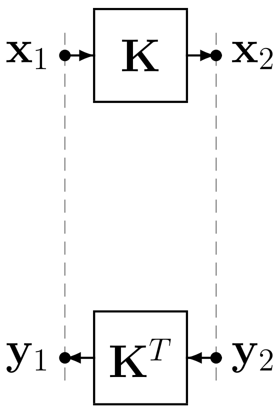
POG variables	Electrical	Mech. Tras.	Mech. Rot.	Hydraulic
Across-var.: v_e	V Voltage	$\dot{x}$ Velocity	ω Angular vel.	P Pressure
Through-var.: v_f	I Current	F Force	τ Torque	Q Volume flow rate

- In each dashed line of the POG schemes the product $P = v_e v_f$ of two power variables v_e and v_f has the physical meaning of *power* P *flowing through that particular power section*.



- The connection blocks convert the power without generating nor dissipating energy.
- The input power flow $\mathbf{x}_1^T \mathbf{y}_1$ is always equal to the output power flow $\mathbf{x}_2^T \mathbf{y}_2$:

$$\begin{aligned}\mathbf{x}_1^T \mathbf{y}_1 &= < \mathbf{x}_1^T, \mathbf{y}_1 > = < \mathbf{x}_1^T, \mathbf{K}^T \mathbf{y}_2 > \\ &= < (\mathbf{K}\mathbf{x}_1)^T, \mathbf{y}_2 > = < \mathbf{x}_2^T, \mathbf{y}_2 > \\ &= \mathbf{x}_2^T \mathbf{y}_2\end{aligned}$$



- The ***power variables*** can be divided in two groups:

  1) the ***"across-variables"*** (voltage V, velocity $\dot{\mathrm{x}}$, angular velocity ω and pressure P) which are defined *"between two points* of the space:

  2) ***"through-variables"*** (current I, force F, torque τ and volume flow rate Q) which are defined *"in each point"* of the space:

Dynamic structure of the energetic domains.

- Each domains is characterized by **only 3 different types** of physical elements:

  2 dynamic elements "$\mathcal{D}_e$" and "$\mathcal{D}_f$" which store the energy (i.e. capacitors, inductors, masses, springs, etc.);

  1 static element "$\mathcal{R}$" which dissipates (or generates) the energy (i.e. resistors, frictions, etc.);

- The dynamics of physical systems can be described using 4 variables:

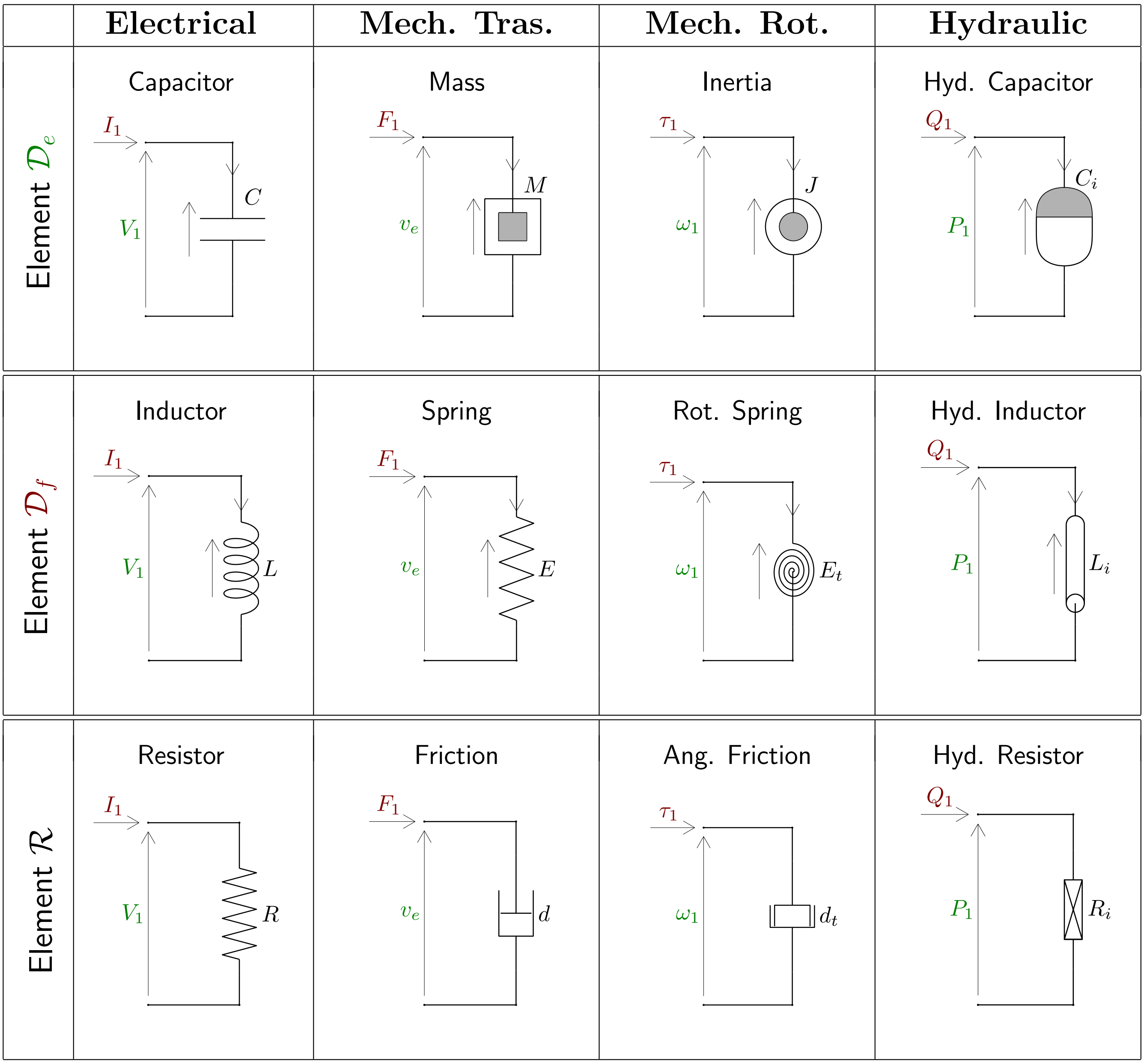
  2 energy variables q_e and q_f which define *how much energy is stored* within the *dynamic elements*;

  2 power variables v_e and v_f which describe *how the energy moves* within the system.

- ***Dynamic structure of the energetic domains:***

	Electrical		**Mech. Tras.**		**Mech. Rot.**		**Hydraulic**	
$\mathcal{D}_e$	C	Capacitor	M	Mass	J	Inertia	C_I	Hyd. Capacitor
q_e	Q	Charge	p	Momentum	p	Ang. Momentum	V	Volume
v_e	V	Voltage	$\dot{x}$	Velocity	ω	Ang. Velocity	P	Pressure
$\mathcal{D}_f$	L	Inductor	E	Spring	E	Spring	L_I	Hyd. Inductor
q_f	ϕ	Flux	x	Displacement	θ	Ang. Displacement	ϕ_I	Hyd. Flux
v_f	I	Current	F	Force	τ	Torque	Q	Volume flow rate
$\mathcal{R}$	R	Resistor	b	Friction	b	Ang. Friction	R_I	Hyd. Resistor

- Graphical representations of the physical elements:

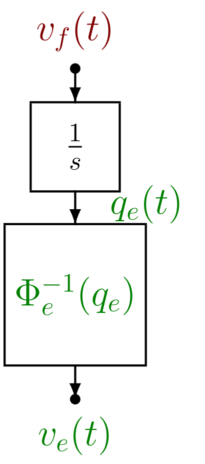


- The ***dynamic element*** D_e is characterized by:

  1) an internal *energy variable* $q_e(t)$;
  2) a *through-variable* $v_f(t)$ as input variable;
  3) an *across-variable* $v_e(t)$ as output variable;
  4) a *constitutive relation* $q_e = \Phi_e(v_e)$ which links the internal energy variable $q_e(t)$ to the output power variable $v_e(t)$;
  5) a *differential equation*

  $$\dot{q}_e(t) = v_f(t)$$

  which links the internal energy variable $q_e(t)$ to the input power variable $v_f(t)$;



Element D_e

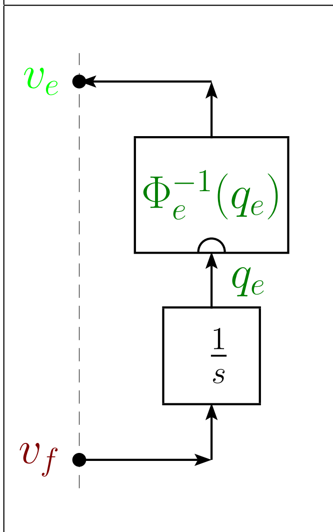
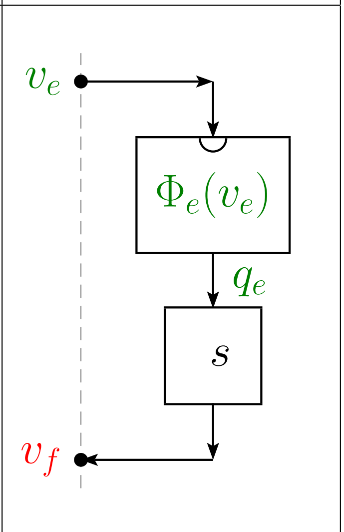
  6) the energy E_e stored in the *dynamic element* D_e is function only of the internal *energy variable* q_e:

  $$E_e = \int_0^t v_e(t)\, v_f(t)\, dt = \int_0^{q_e} \Phi_e^{-1}(q_e)\, dq_e = E_e(q_e).$$

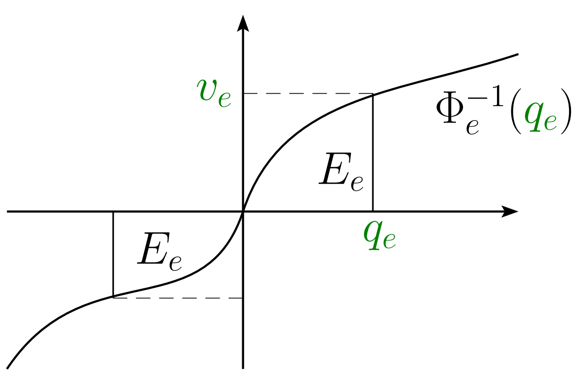
  where the following substitutions have been used:

  $$v_e(t) = \Phi_e^{-1}(q_e) \qquad\qquad dq_e = v_f(t)\, dt$$

- Dynamic orientation and stored energy:

1) Integral	2) Derivative
	
TO BE USED	DO NOT USE

Stored energy E_e

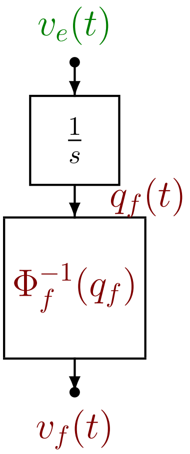


- The ***dynamic element*** D_f has a structure which is *"dual"* respect to the structure of *dynamic element* D_e.

  1) an internal *energy variable* $q_f(t)$;
  2) an *across-variable* $v_e(t)$ as input variable;
  3) a *through-variable* $v_f(t)$ as output variable;
  4) a *constitutive relation* $q_f = \Phi_f(v_f)$ which links the internal energy variable $q_f(t)$ to the output power variable $v_f(t)$;
  5) a *differential equation*

  $$\dot{q}_f(t) = v_e(t)$$

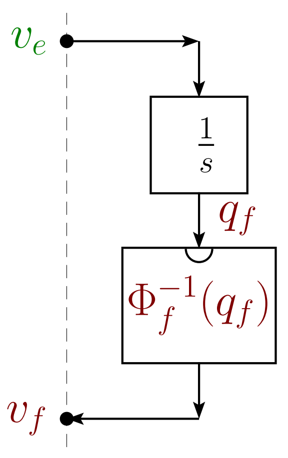
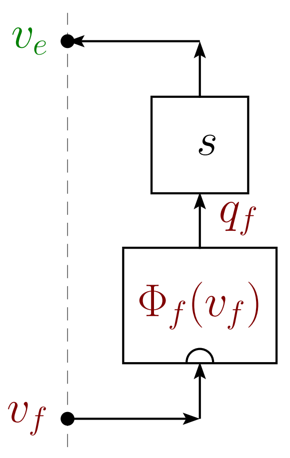
  which links the internal energy variable $q_f(t)$ to the input power variable $v_e(t)$;



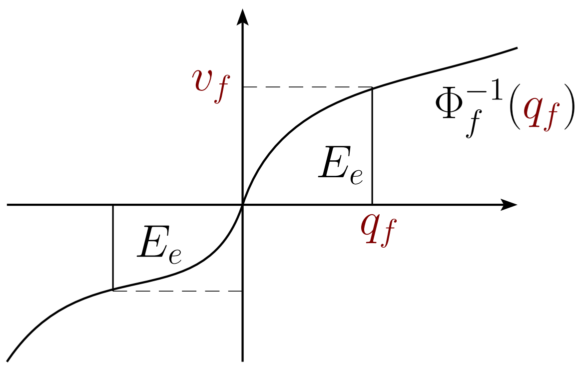
Element D_f

The dual structure can be easily obtained performing the following substitutions: $q_e(t) \to q_f(t)$, $v_f(t) \leftrightarrow v_e(t)$ and $\Phi_e(v_e) \to \Phi_f(v_f)$.

- Dynamic orientation and stored energy:

1) Integral	2) Derivative
	
TO BE USED	DO NOT USE

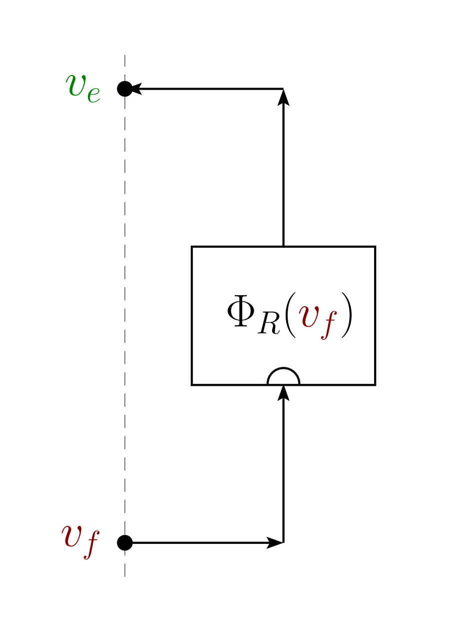
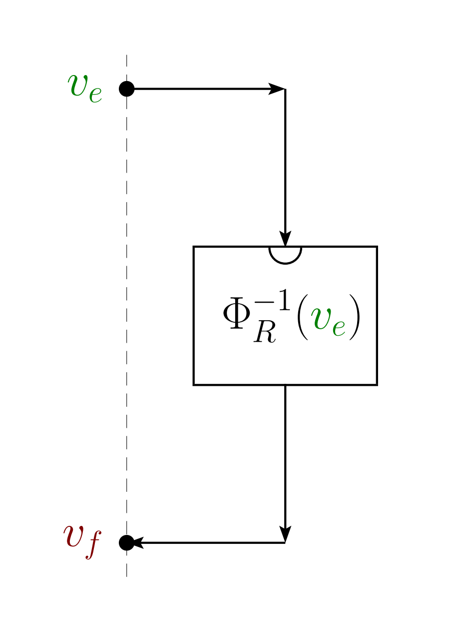
Stored energy E_e



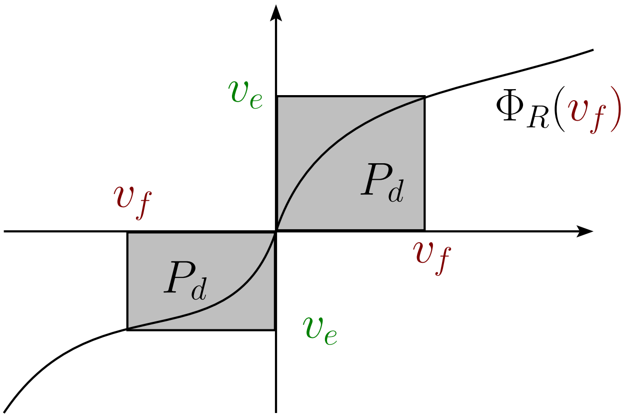
- Note: the energy variables q_e and q_f are the integral of the input power variables $v_f(t)$ and $v_e(t)$:

$$q_e = \int_0^t v_f(t)\,dt, \qquad q_f = \int_0^t v_e(t)\,dt.$$

- The *static element* $\mathcal{R}$ is completely characterized by a static function $v_e = \Phi_R(v_f)$ which links the input variable v_f to the output variable v_e.

1) Resistance	2) Conductance
	

- Dissipated power P_d of the *static element* $\mathcal{R}$:



- The differential equation of a physical element can be obtained imposing the time-derivative of the energy variable equal to the input power variable:

1) For $\mathcal{D}_f$ elements: $\dot{q}_f(t) = v_e(t) \quad \Leftrightarrow \quad \dfrac{d\,\dot{q}_f(t)}{dt} = v_e(t)$

2) For $\mathcal{D}_e$ elements: $\dot{q}_e(t) = v_f(t) \quad \Leftrightarrow \quad \dfrac{d\,\dot{q}_e(t)}{dt} = v_f(t)$

- **Electromagnetic domain**:

	Name	Constitutive Rel.	Linear case	Differentioal Eq.
$\mathcal{D}_e$ q_e v_e	C Capacitor Q Charge V Voltage	$Q = \Phi_C(V)$	$Q = C\,V$	$\frac{d\,Q}{dt} = I$
$\mathcal{D}_f$ q_f v_f	L Inductor ϕ Flux I Current	$\phi = \Phi_L(I)$	$\phi = L\,I$	$\frac{d\,\phi}{dt} = V$
$\mathcal{R}$	R Resistance	$V = \Phi_R(I)$	$V = R\,I$	

- **Mechanic Translational domain:**

	Name	Constitutive Rel.	Linear case	Differentioal Eq.
$\mathcal{D}_e$ q_e v_e	M Mass P Momentum $\dot{x}$ Velocity	$P = \Phi_M(\dot{x})$	$P = M\,\dot{x}$	$\frac{d\,P}{dt} = F$
$\mathcal{D}_f$ q_f v_f	E String x Displacement F Force	$x = \Phi_E(F)$	$x = E\,F$	$\frac{d\,x}{dt} = \dot{x}$
$\mathcal{R}$	b Friction	$F = \Phi_b(\dot{x})$	$F = b\,\dot{x}$	

- **Mechanic Rotational domain**:

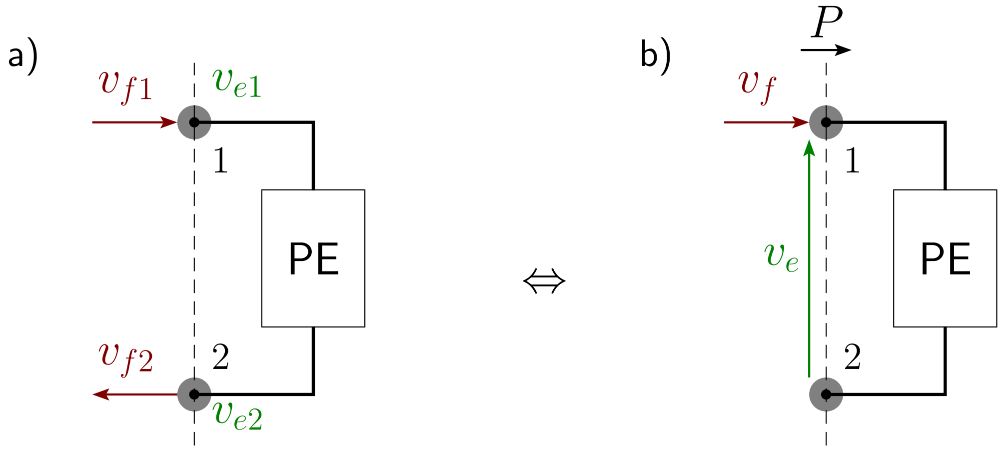
	Name	Constitutive Rel.	Linear case	Differentioal Eq.
$\mathcal{D}_e$ q_e v_e	J Inertia P Ang. Momentum ω Ang. Velocity	$P = \Phi_J(\omega)$	$P = J\,\omega$	$\frac{d\,P}{dt} = \tau$
$\mathcal{D}_f$ q_f v_f	E Rot. Spring θ Ang. Displacement τ Torque	$\theta = \Phi_E(\tau)$	$\theta = E\,\tau$	$\frac{d\,\theta}{dt} = \omega$
$\mathcal{R}$	b Rot. Friction	$\tau = \Phi_b(\omega)$	$\tau = b\,\omega$	

- **Hydraulic domain**:

	Name	Constitutive Rel.	Linear case	Differentioal Eq.
$\mathcal{D}_e$ q_e v_e	C_I Hyd. Capacitor V Volume P Pressure	$V = \Phi_C(P)$	$V = C_I\,P$	$\frac{d\,V}{dt} = Q$
$\mathcal{D}_f$ q_f v_f	L_I Hyd. Inductor ϕ_I Hyd. Flux Q Volume flow rate	$\phi_I = \Phi_L(Q)$	$\phi_I = L_I\,Q$	$\frac{d\,\phi_I}{dt} = P$
$\mathcal{R}$	R Hyd. Resistor	$P = \Phi_R(Q)$	$P = R_I\,Q$	

Connection of physical elements

- ***Physical Elements***. The physical systems are composed by **physical elements** (PE) (i.e. *dynamic elements* D_e and D_f or *static element* $\mathcal{R}$) which interact with the external world by means of two terminals:



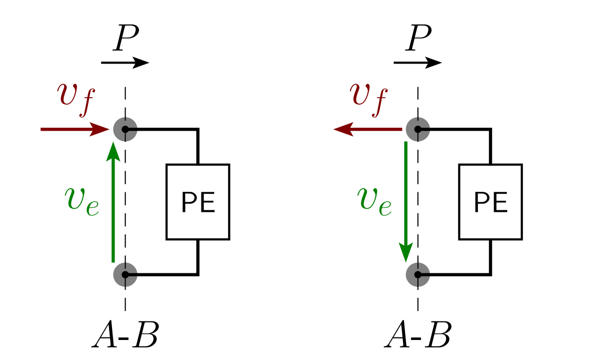
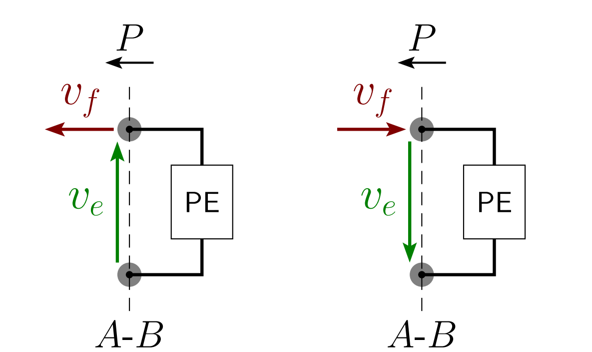
Each terminal, see case a), **is characterized by two power variables** (v_{e1}, v_{f1}) and (v_{e2}, v_{f2}). Choosing $v_e = v_{e1} - v_{e2}$ and $v_f = v_{f1} = v_{f2}$ as new power variables, the power interaction of the PE with the external world can be described using the power section P in case b).

- The value of the power P flowing through the section is the product of the two power variables $v_e(t)$ and $v_f(t)$:

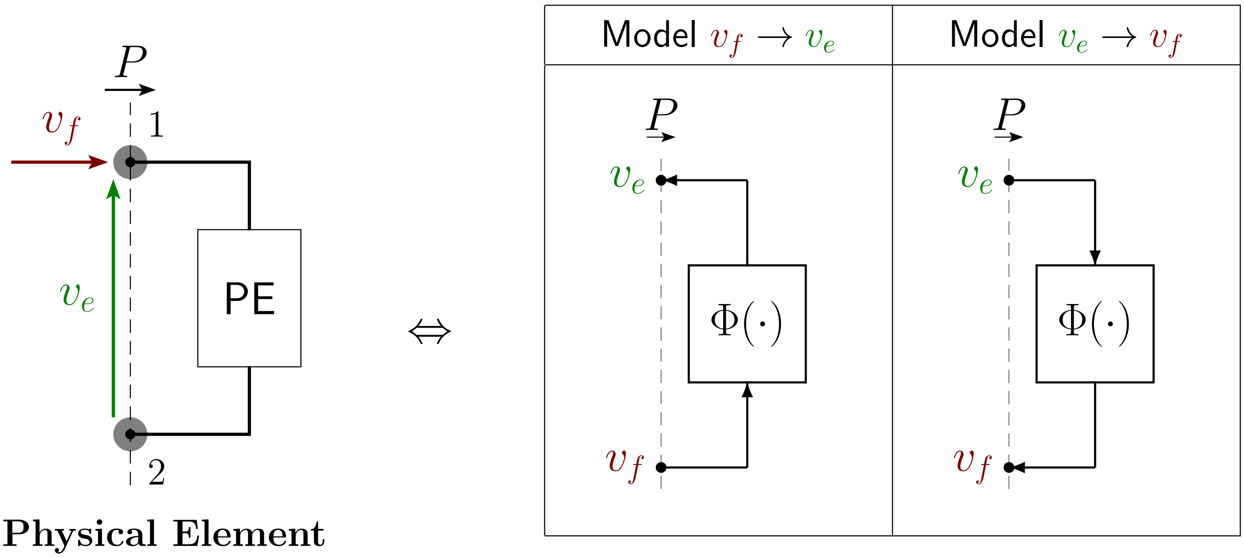
$$P(t) = v_e(t)\, v_f(t)$$

The sign and the direction of power $P(t)$ depend on the sign and the reference positive direction chosen for the variables $v_e(t)$ and $v_f(t)$.

- The **signs of the power** P flowing through a physical section A-B are:

	a) Power P flows from A to B	b) Power P flows from B to A
Power flows		

- ***Integral and derivative causality***. The POG dynamic model of a physical elements (PE), that is an element D_e, D_f or $\mathcal{R}$, can be graphically described by using two block schemes having different orientation:



Physical Element

The two possible "orientations" of the PE dynamic model are:

1) v_f as input and v_e as output: model $v_f \to v_e$
2) v_e as input and v_f as output: model $v_e \to v_f$.

The function $\Phi(\cdot)$ shown in the figures symbolically represents the dynamic or the static equation describing the physical element.

- If PE is a static element $\mathcal{R}$, the two diagrams are both suitable for describing the mathematical model of the physical element.

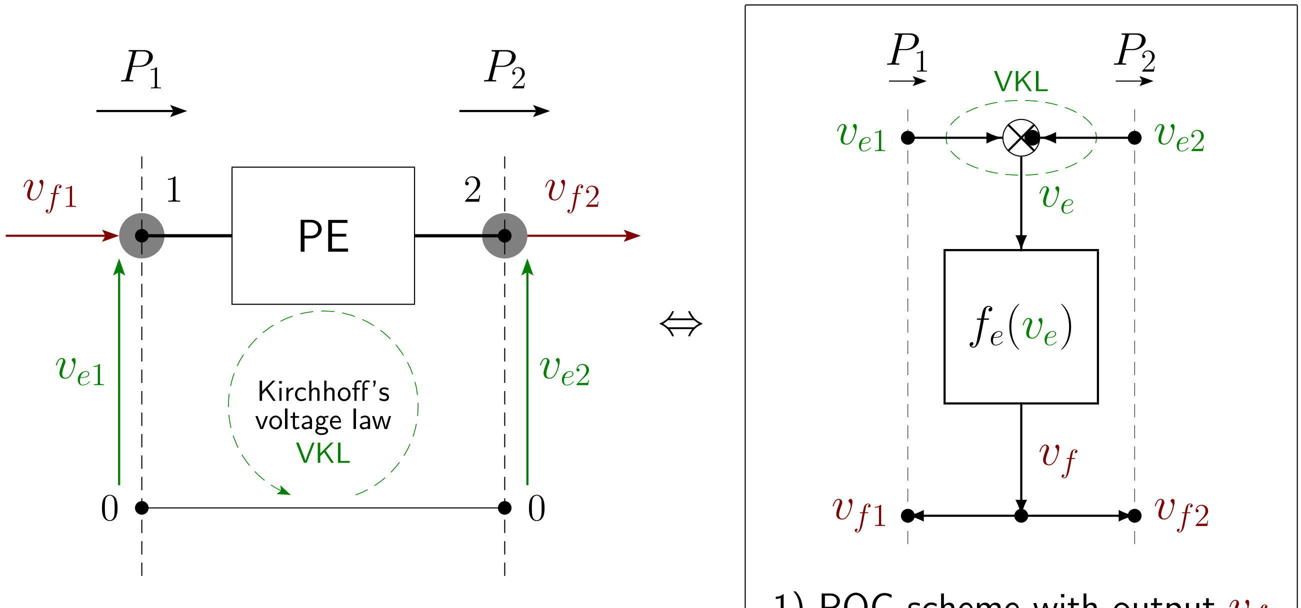
- If PE is a dynamic element D_e or D_f, the two diagrams represent the two possible **causality modes** of the physical element:

  1) the ***integral causality*** (TO BE USED) is physically realizable, useful in simulation and is the preferred dynamic model in the POG technique.

  2) the ***derivative causality*** (DO NOT USE) is still a correct mathematical model of the PE, but it is not used in the POG technique because it is not physically realizable and it is not useful in simulation.

- Each Physical Element (PE) interacts with the external world through the power sections associated to its terminals. Two basic connections are possible: ***series*** and ***parallel***.

- <u>***Series***</u>: a Physical Element PE is connected **in series** if its terminals share the same through-variable $v_f = v_{f1} = v_{f2}$:

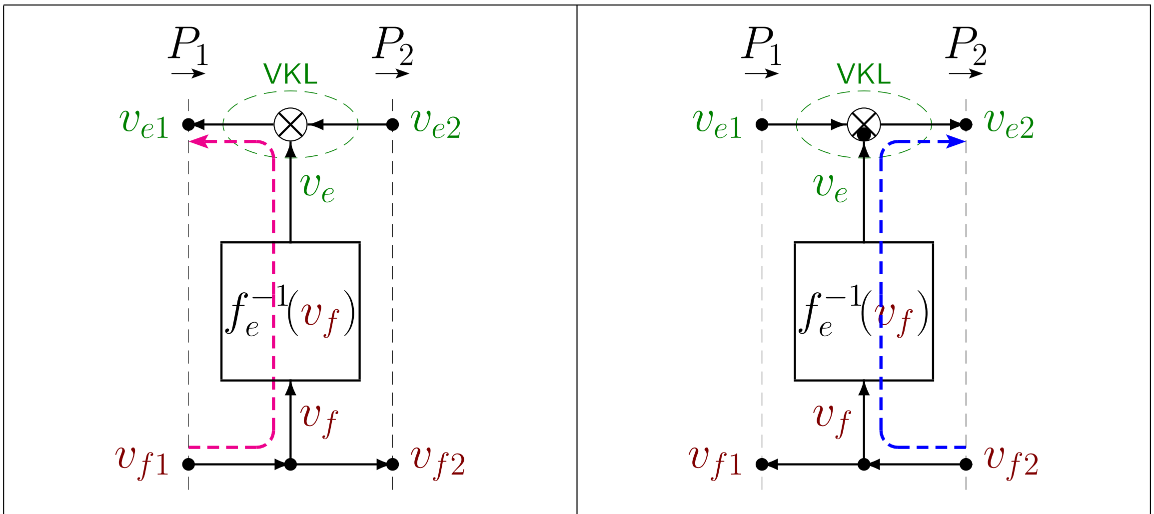


Physical Element connected in series

1) POG scheme with output v_f

- The summation element is a mathematical description of the **Voltage Kirchhoff's Law** (VKL) applied to a "closed" path which involves the *across variables* v_{e1}, v_{e2} and v_e

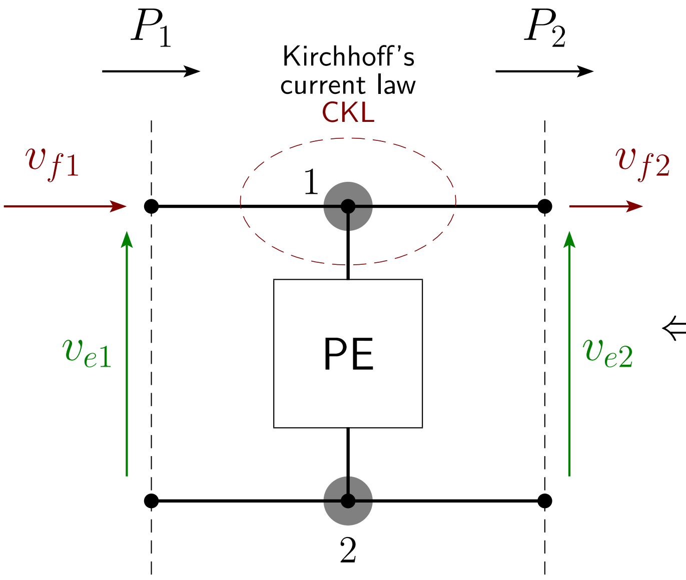
- Inverting the input and output paths of the POG block scheme 1) one obtains the following equivalent POG block schemes:



2) POG scheme with output v_{e1}.

3) POG scheme with output v_{e2}.

Parallel: a Physical Element PE is connected **in parallel** if its terminals share the same across-variable $v_e = v_{e1} = v_{e2}$:



Physical Element connected in parallel

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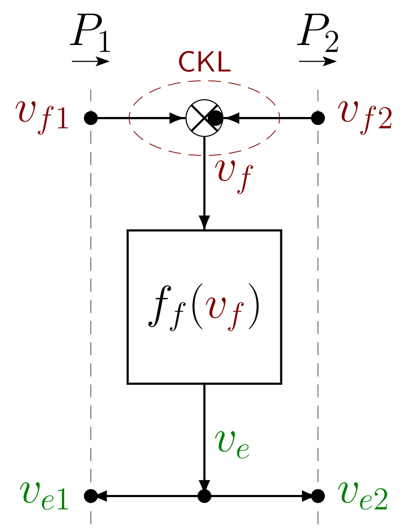
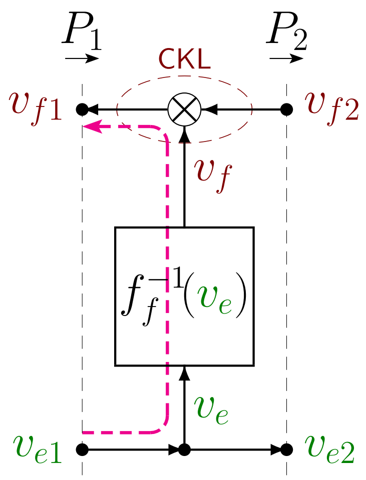
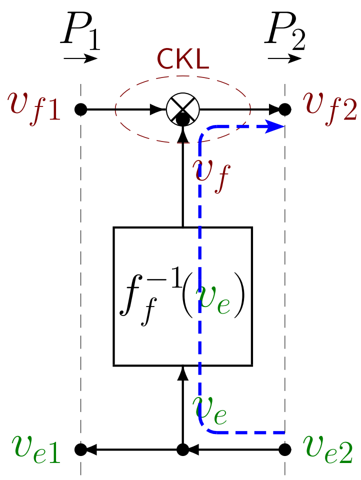


1) POG scheme with output v_e.

- The summation element is a mathematical description of the **Current Kirchhoff's Law** (CKL) applied to a node which involves the *through variables* v_{f1}, v_{f2} and v_f.
- Inverting the input and output paths of the POG block scheme 1) one obtains the following equivalent POG block schemes:



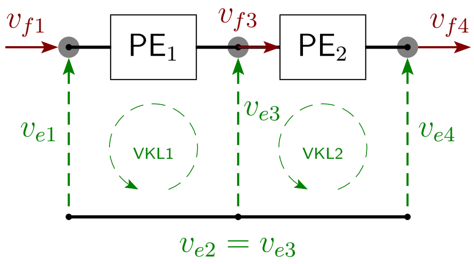
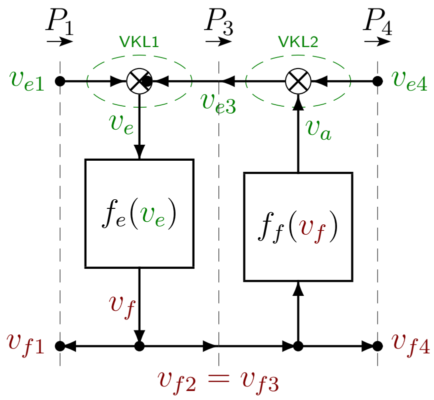
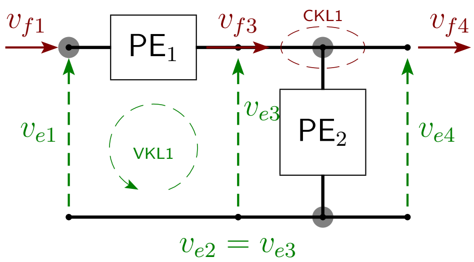
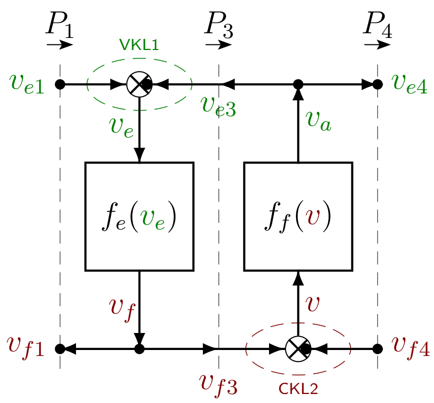
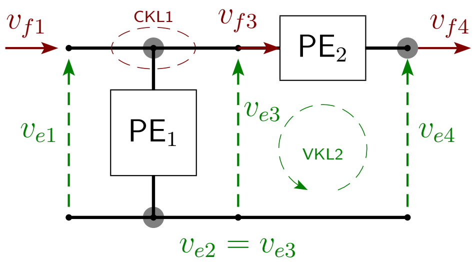
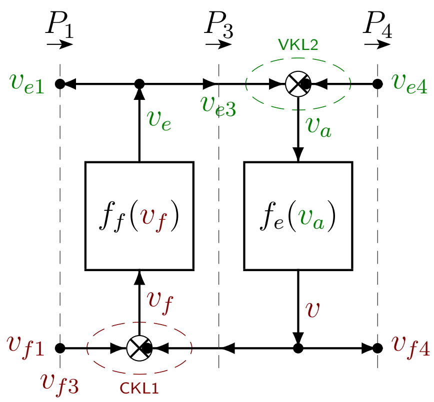
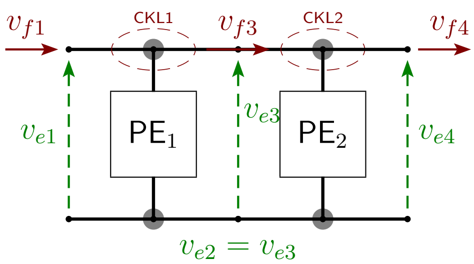
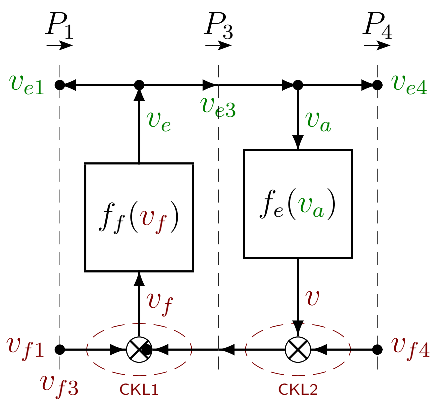
2) POG scheme with output v_{f1}.



3) POG scheme with output v_{f2}.

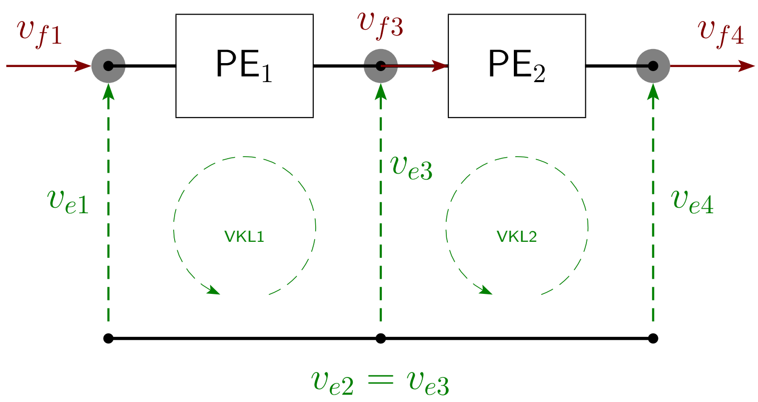
Connecting physical elements

- Two physical elements PE_1 and PE_2 can be connected as follows:

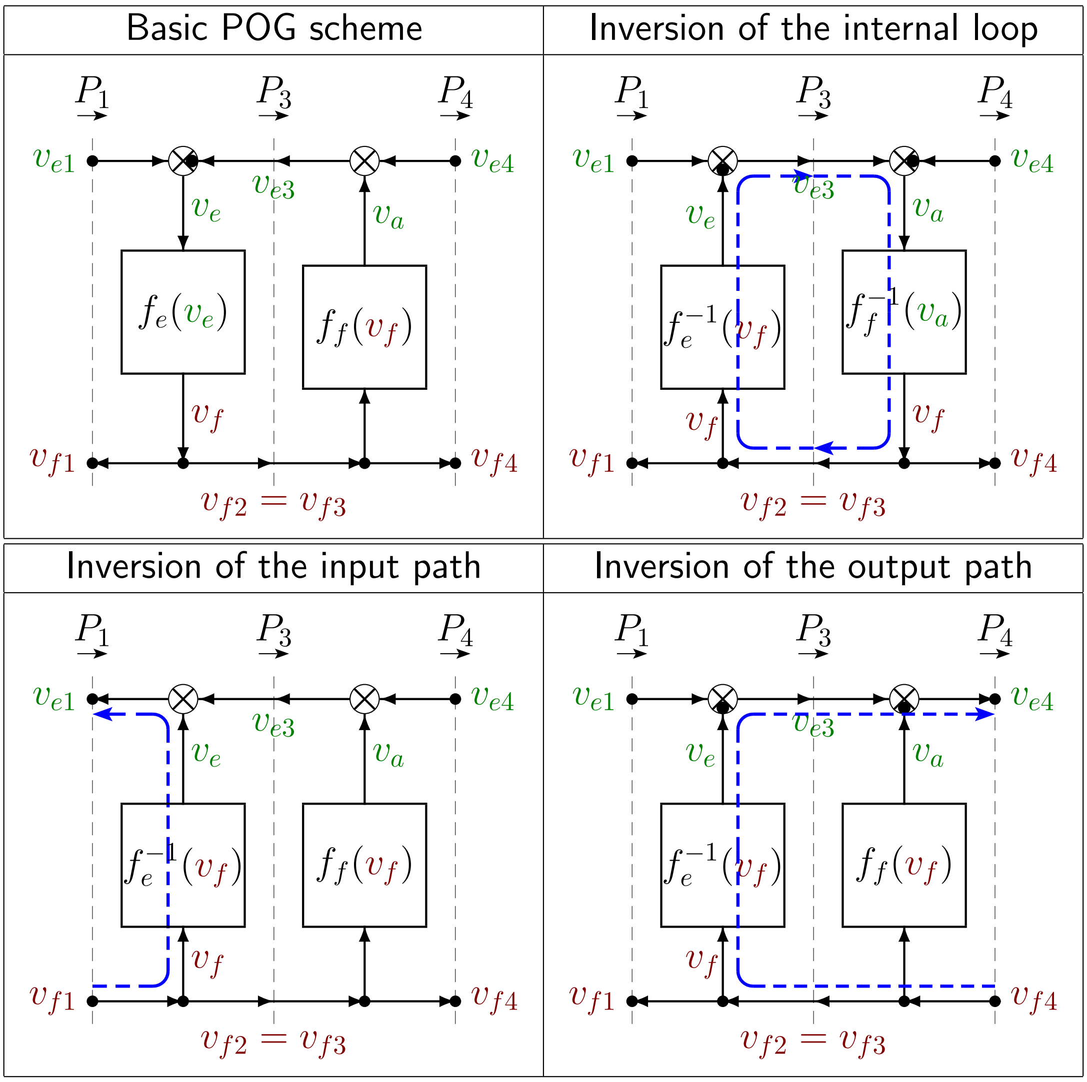
Physical connection	Basic POG scheme
a) *Series - Series* 	
b) *Series - Parallel* 	
c) *Parallel - Series* 	
d) *Parallel - Parallel* 	

- The basic POG scheme associated to a PE_1-PE_2 connection can be drawn in four different ways. For the "*Series - Series*" connection:

a) *Series - Series*

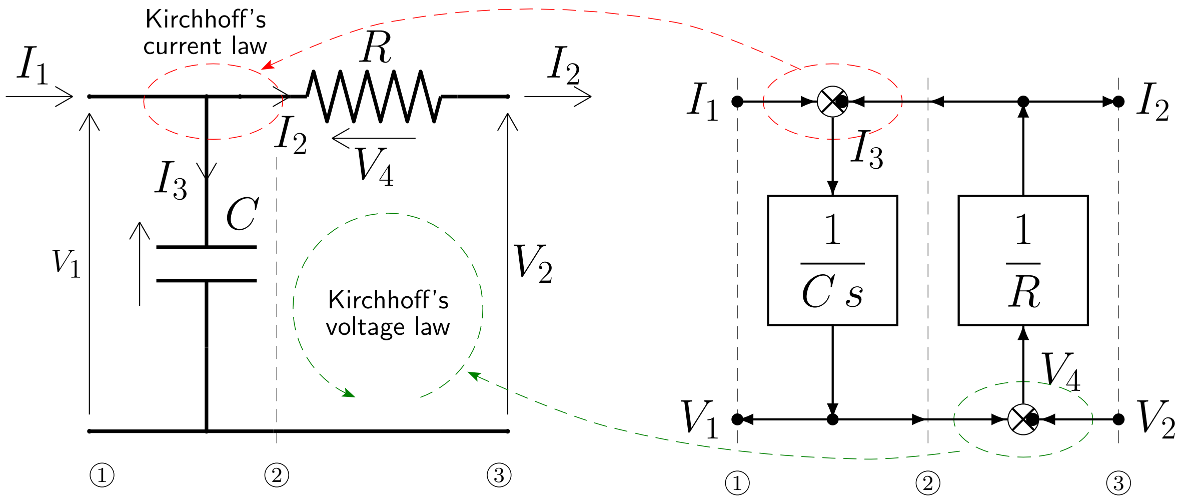


the following POG schemes can be used:



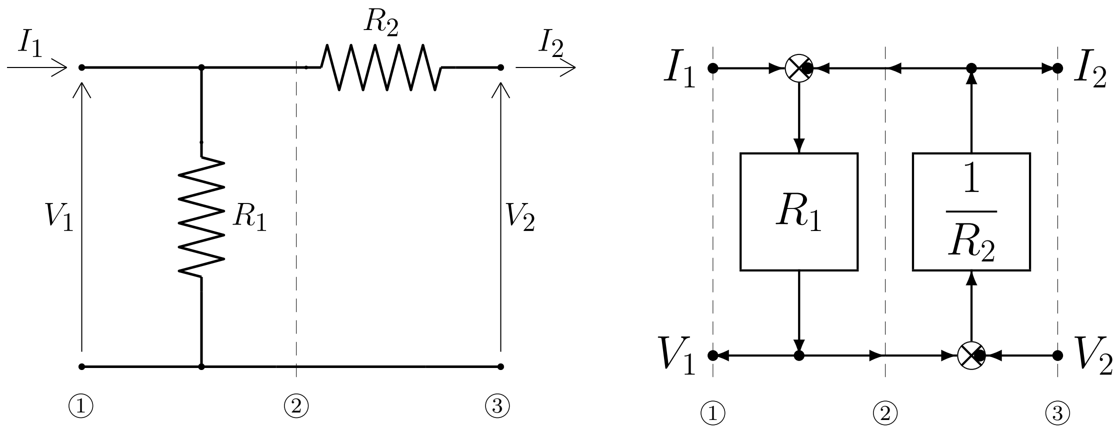
- Other four possible POG block schemes can be obtained considering the "upside down" versions of the above reported POG schemes.

- Example. A C-parallel and R-series connection:

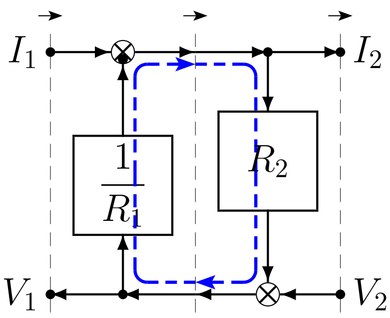
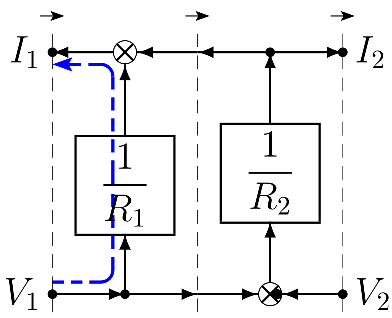
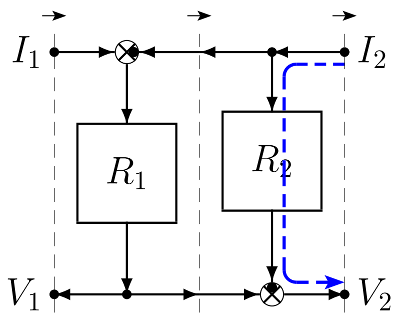


The internal loop and the input path of the POG scheme CANNOT be inverted because the capacitor must be described using its "integral causality" model. The output path can be inverted.

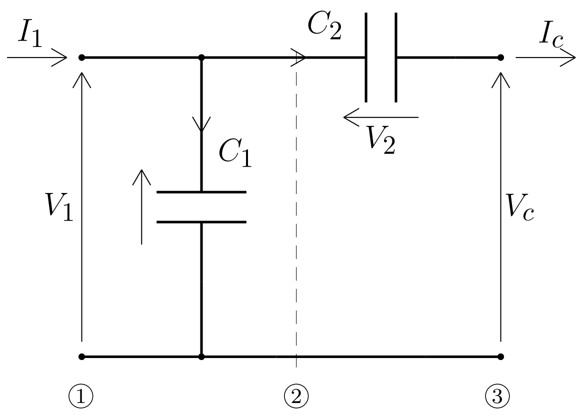
- Example. A R_1-parallel and R_2-series connection:

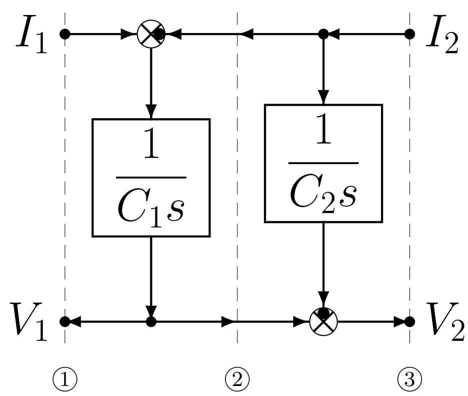


The following equivalent block schemes can be used:

Inversion of the internal loop	Inversion of the input path	Inversion of the output path
		

- Example. A C_1-parallel and C_2-series connection:



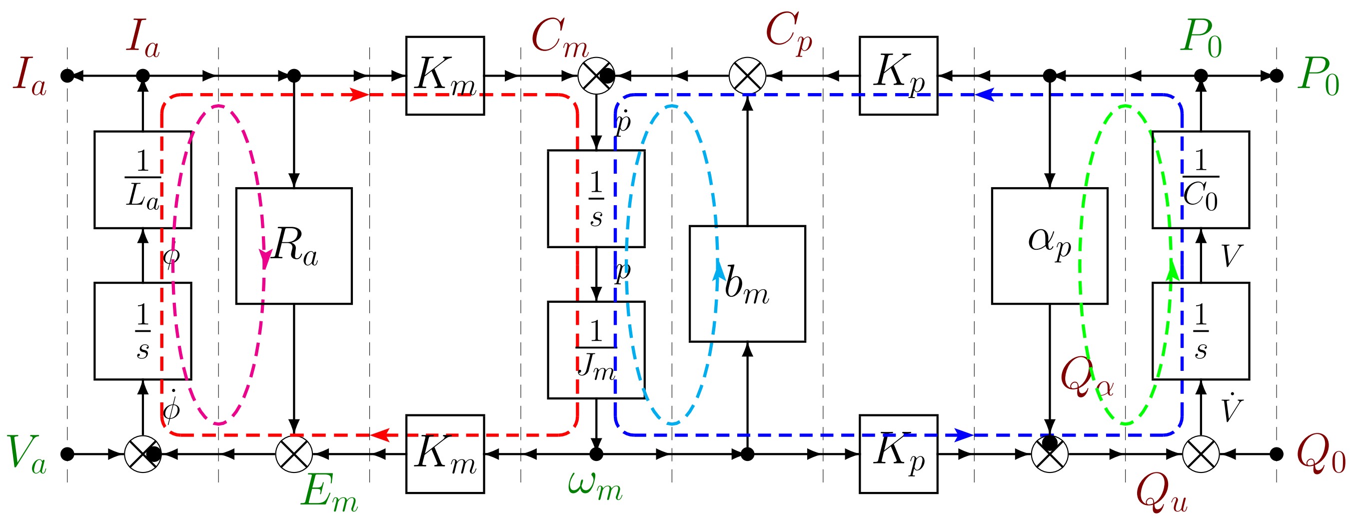


In this case there is only one POG block scheme that can be used for describing the given physical system.

- Often, when two physical elements are connected, *a feedback loop appears* in the corresponding POG block scheme. The following property holds.

- ***Property***. ***All the loops*** of a POG scheme ***contains an odd number of minus signs*** (i.e. black spots in the summation elements).

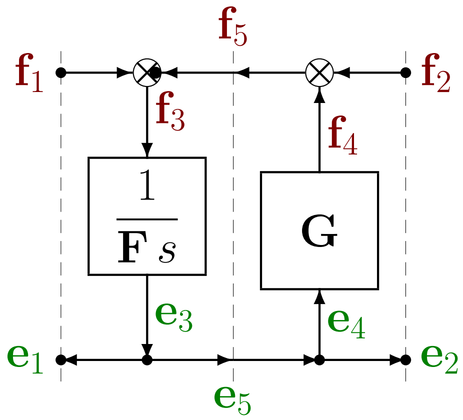
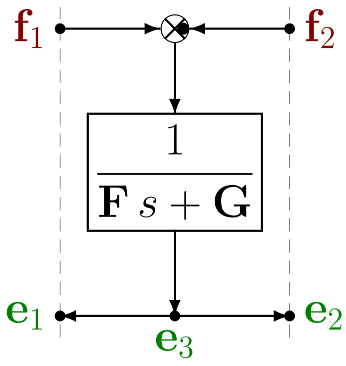
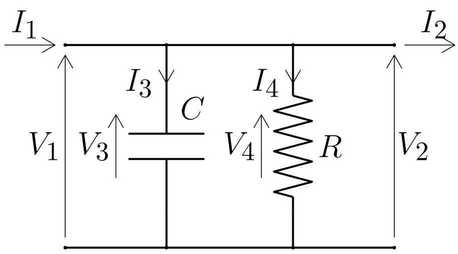
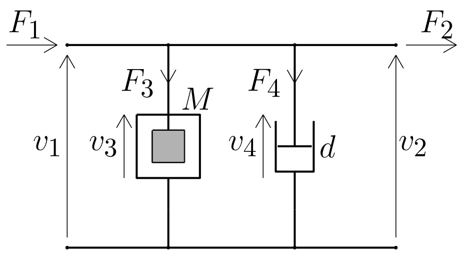
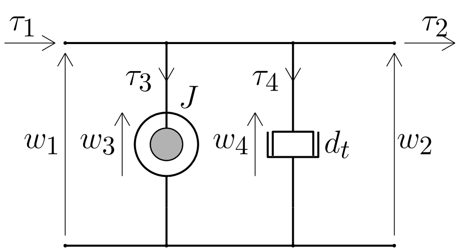
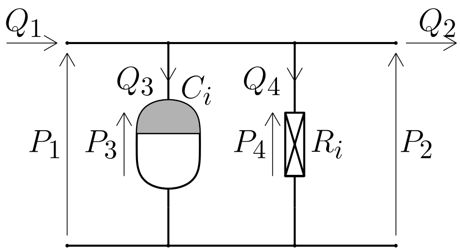
  Example:



All the five loops of this block scheme contain "one" minus sign.

- This rule can be used to verify the "consistency" and the correctness of the considered POG block scheme.

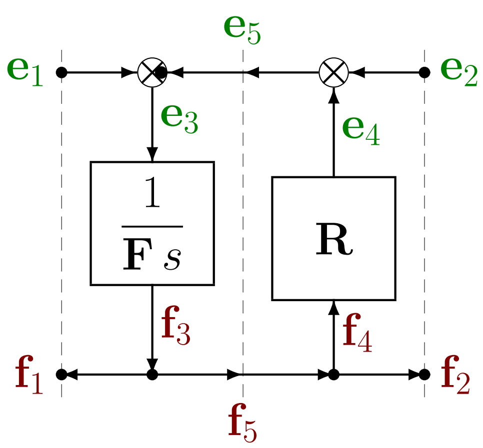
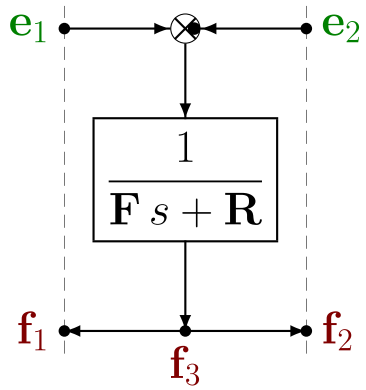
- Examples of a "*Parallel - Parallel*" connection.

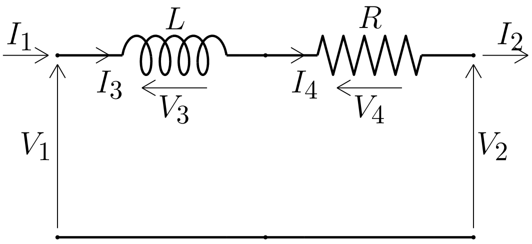
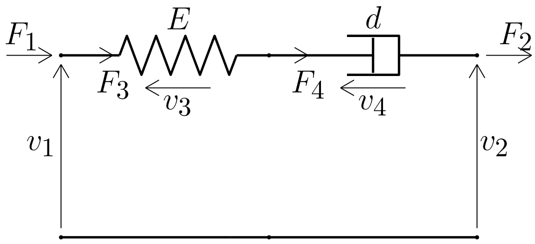
Basic POG scheme	Reduced POG scheme
	
Electrical	**Mech. Traslational**
	
$\mathbf{f}=I,\ \ \mathbf{F}=C,\ \ \mathbf{e}=V,\ \ \mathbf{G}=\frac{1}{R}$	$\mathbf{f}=F,\ \ \mathbf{F}=M,\ \ \mathbf{e}=v,\ \ \mathbf{G}=\frac{1}{d}$
Mech. Rotational	**Hydraulic**
	
$\mathbf{f}=\tau,\ \ \mathbf{F}=J,\ \ \mathbf{e}=w,\ \ \mathbf{G}=\frac{1}{d_t}$	$\mathbf{f}=Q,\ \ \mathbf{F}=C_i,\ \ \mathbf{e}=P,\ \ \mathbf{G}=\frac{1}{R_i}$

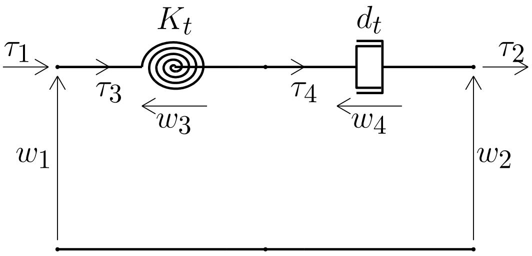
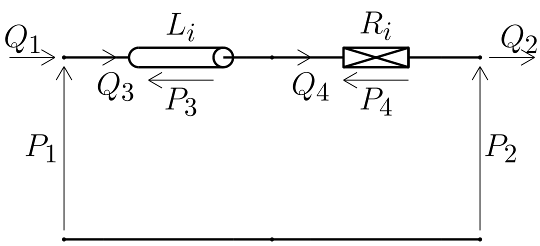
The reduced model is obtained using the Mason formula:

$$G_0(s)=\frac{\dfrac{1}{\mathbf{F}s}}{1+\dfrac{\mathbf{G}}{\mathbf{F}s}}=\frac{1}{\mathbf{F}s+\mathbf{G}}$$

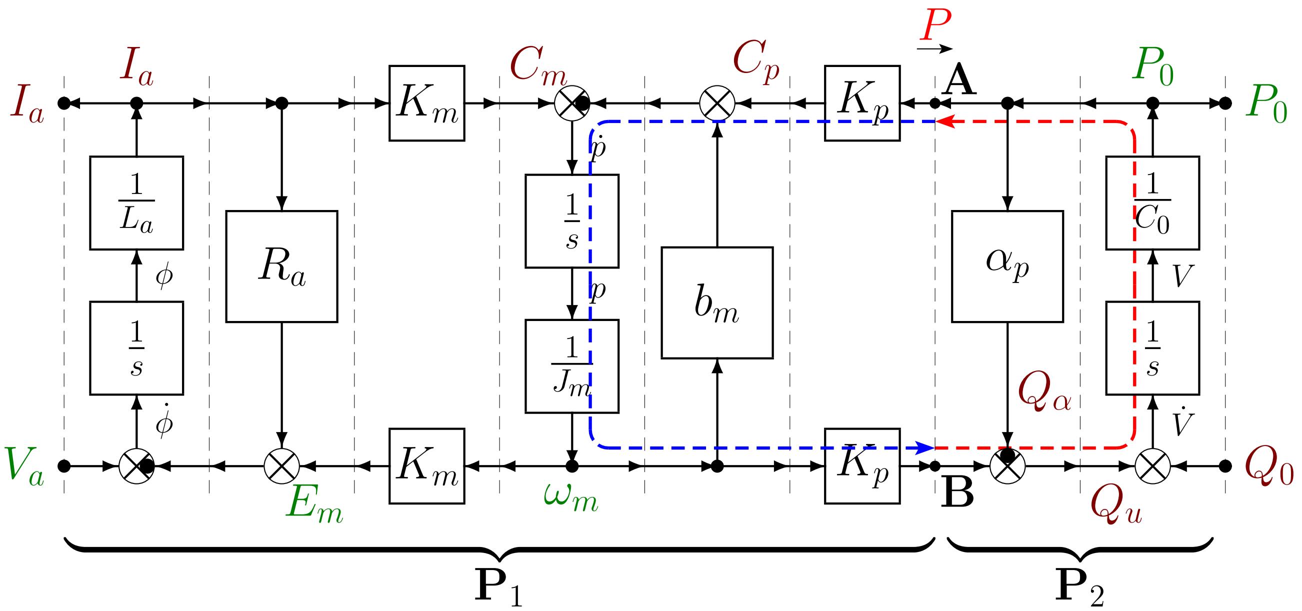
- Examples of a "*Series* - *Series*" connection.

Basic POG scheme	Reduced POG scheme
	

Electrical	Mech. Traslational
	
$\mathbf{e} = V,\ \mathbf{F} = L,\ \mathbf{f} = I,\ \mathbf{R} = R$	$\mathbf{e} = v,\ \mathbf{F} = E,\ \mathbf{f} = F,\ \mathbf{R} = d$

Mech. Rotational	Hydraulic
	
$\mathbf{e} = w,\ \mathbf{F} = E_t,\ \mathbf{f} = \tau,\ \mathbf{R} = d_t$	$\mathbf{e} = P,\ \mathbf{F} = L_i,\ \mathbf{f} = Q,\ \mathbf{R} = R_i$

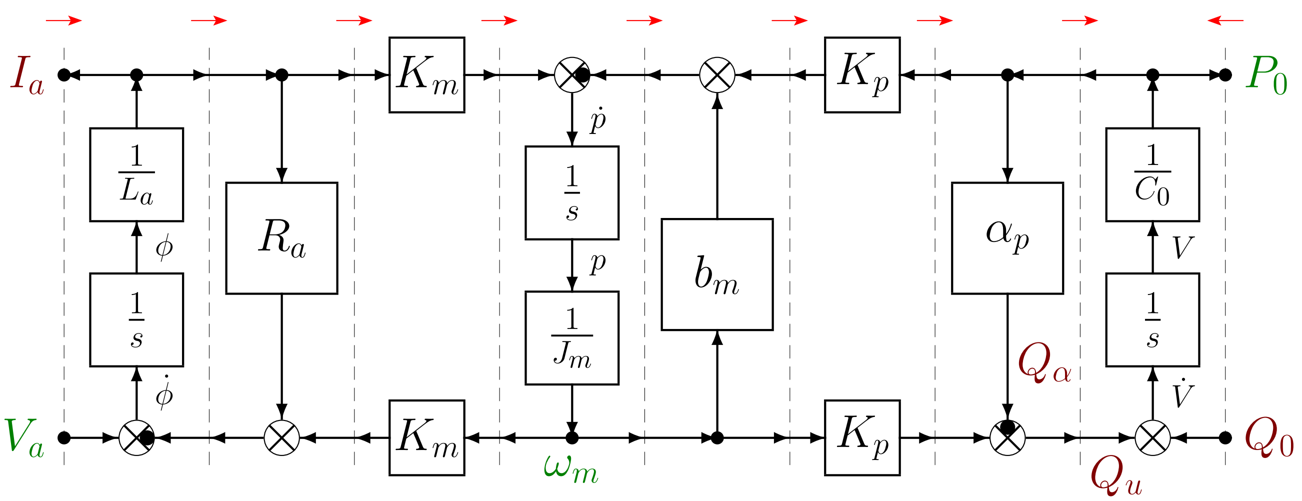
- **Property**. The ***direction*** of the power P flowing through a section of a POG block scheme ***is "positive"*** if an ***"even" number of minus signs*** is present along the paths which goes from the input to the output.



Let us consider, for example, the power section **A**-**B** which divides the block scheme in two sections: $\mathbf{P}_1$ and $\mathbf{P}_2$. The power P flows from section $\mathbf{P}_1$ to section $\mathbf{P}_2$ because:

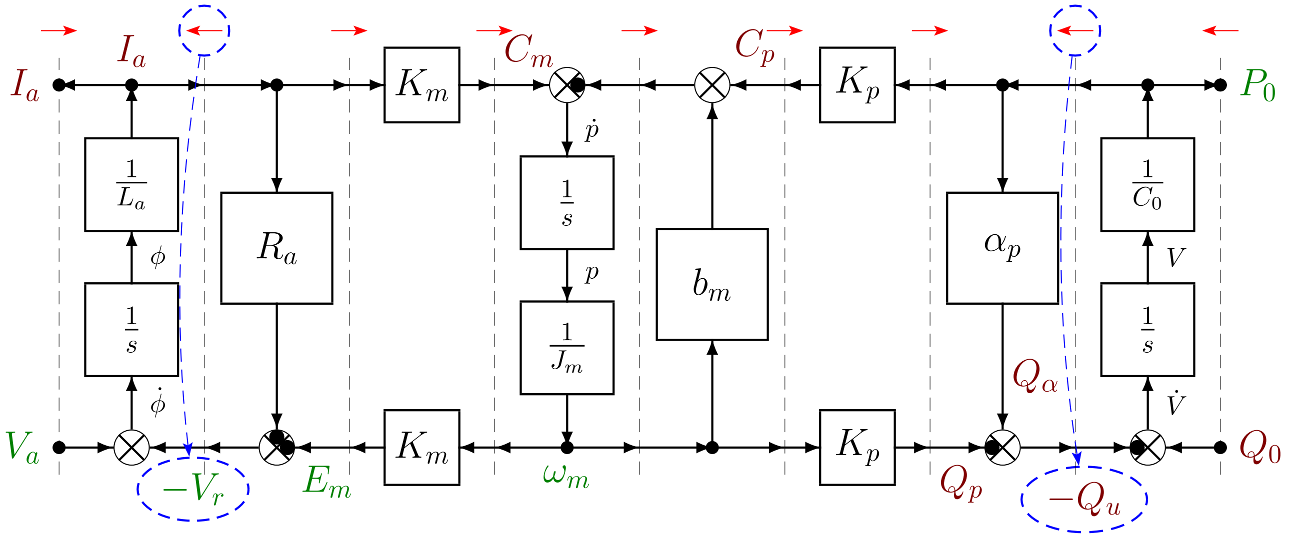
- the red dashed path that goes from **B** to **A** within section $\mathbf{P}_2$ contains "zero" minus signs (i.e. an even number);
- the blue dashed path that goes from **A** to **B** within section $\mathbf{P}_1$ contains "one" minus signs (i.e. an odd number);

- Using the previous rule it is possible to compute the positive direction of the poter flows in each power section of a POG block schemes:

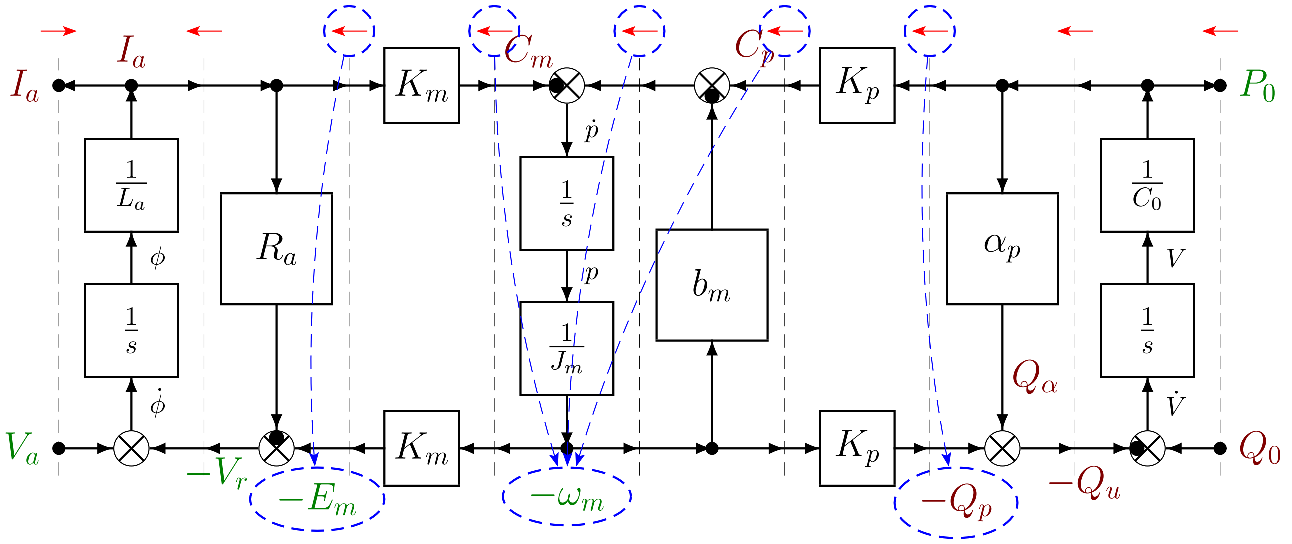


- The signs of the power flows depend on the sign of the power variables.

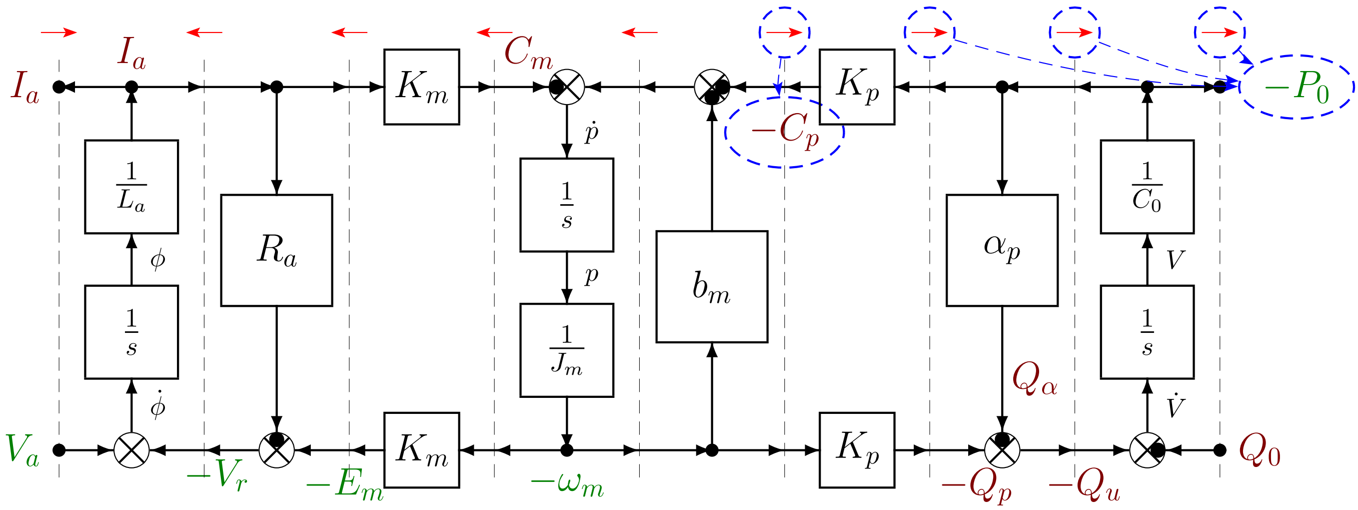
a) Changing the signs of variables V_r and Q_u one obtains:



b) Changing the signs of variables E_m, ω_m and Q_p one obtains:

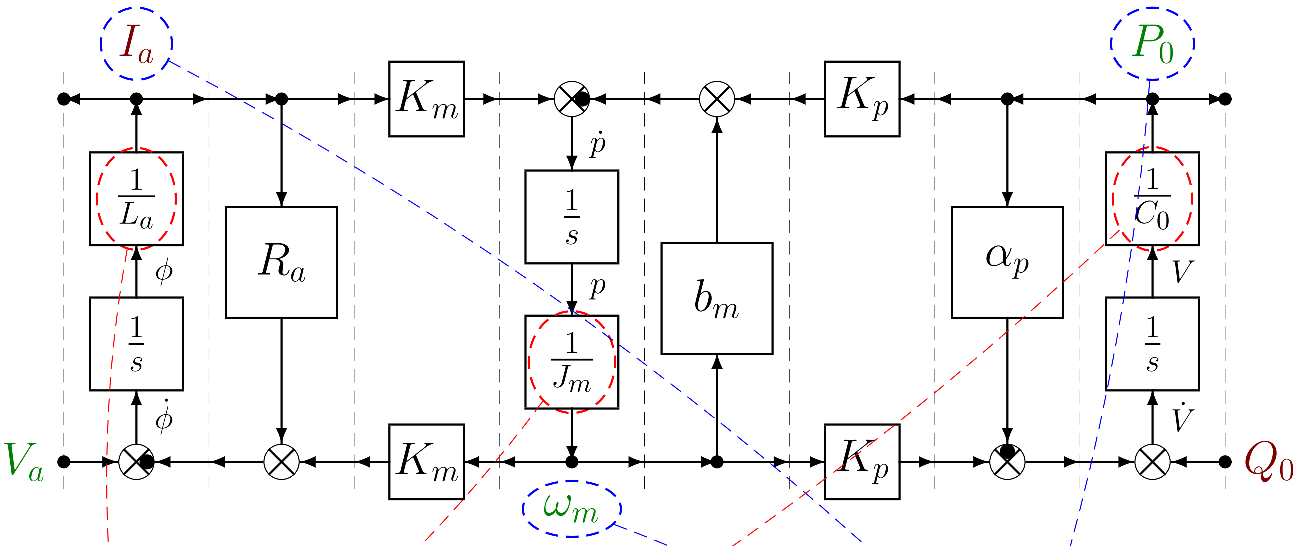


c) Changing the signs of variables C_p and P_0 one obtains:



From POG scheme to State space model

- From a POG block scheme:



one can directly obtain the corresponding POG state space model:

$$\underbrace{\begin{bmatrix} L_a & 0 & 0 \\ 0 & J_m & 0 \\ 0 & 0 & C_0 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_a \\ \dot{\omega}_m \\ \dot{P}_0 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_a & -K_m & 0 \\ K_m & -b_m & -K_p \\ 0 & K_p & -\alpha_p \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_a \\ \omega_m \\ P_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ Q_0 \end{bmatrix}}_{\mathbf{u}}$$

$$\mathbf{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \mathbf{u}$$

- The following procedure must be used:

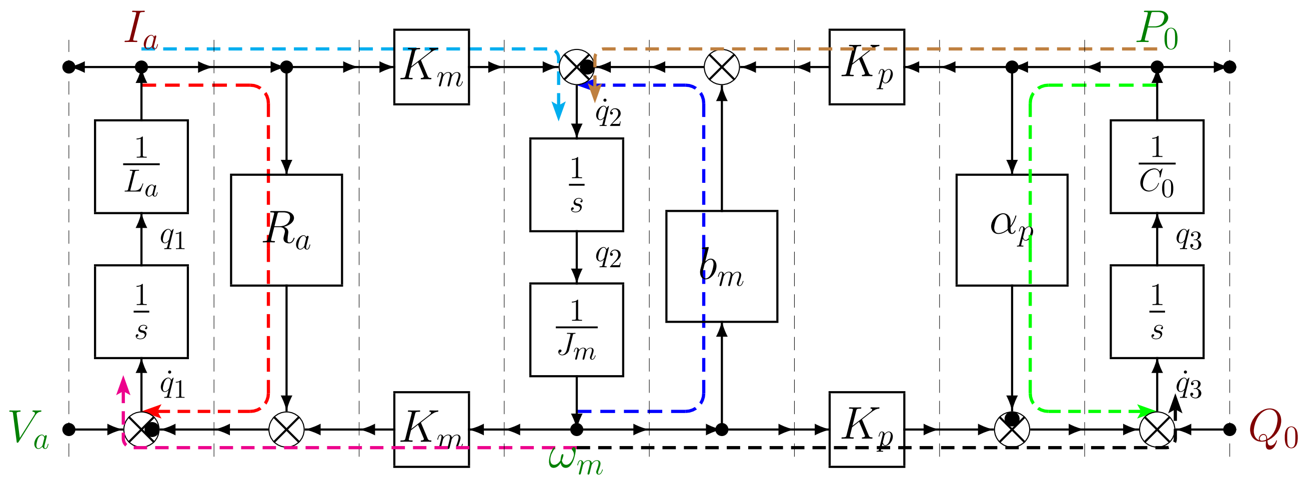
1) The components of the state vector $\mathbf{x}$ must be chosen equal to the "*output power variables*" of the dynamic elements:

$$\mathbf{x} = \begin{bmatrix} I_a & \omega_m & V_0 \end{bmatrix}^T$$

2) The coefficients L_a, J_m and C_0 of the diagonal matrix $\mathbf{L}$ (i.e. the energy matrix) are the coefficients that links the "*output power variables*" to the "*internal energy variables*" within the *constitutive relation*:

$$\begin{aligned} \phi &= L_a I_a \\ p &= J_m \omega_m \\ V &= C_0 P_0 \end{aligned} \quad \Rightarrow \quad \mathbf{L} = \begin{bmatrix} L_a & 0 & 0 \\ 0 & J_m & 0 \\ 0 & 0 & C_0 \end{bmatrix}$$

3) The coefficients A_{ij} of matrix $\mathbf{A}$ are the gains of all the paths that link the j-th state variables x_j to the i-th input $\dot{q}_i$ of the integrators.



$$\underbrace{\begin{bmatrix} L_a & 0 & 0 \\ 0 & J_m & 0 \\ 0 & 0 & C_0 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{I}_a \\ \dot{\omega}_m \\ \dot{P}_0 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -R_a & -K_m & 0 \\ K_m & -b_m & -K_p \\ 0 & K_p & -\alpha_p \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_a \\ \omega_m \\ P_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_a \\ Q_0 \end{bmatrix}}_{\mathbf{u}}$$

$$\mathbf{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \mathbf{u}$$

The element $A_{23} = -K_p$ of matrix $\mathbf{A}$, for example, is the gain of the path that links the third state variable $x_3 = P_0$ to the input $\dot{q}_2 = \dot{p}$ of the second integrator.

4) The coefficients B_{ij}, C_{ij} and D_{ij} of matrices $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, respectively, can be determined in a similar way.

Coefficients "B_{ij}" are the gains of the paths that link the **j-th input** u_j to the **i-th input $\dot{q}_i$ of the integrators**. The coefficient $B_{32} = 1$, for example, is the gain of the path that goes from input $u_2 = Q_0$ to the input $\dot{q}_3$ of the third integrator.

Coefficients "C_{ij}" are the gains of the paths that link the **j-th state variable** x_j to the **i-th output** y_i of the system;

Coefficients "D_{ij}" are the gains of the paths that link the **j-th input** u_j to the **i-th output** y_i of the system;

Properties of a linear POG state space model

1) The energy matrix $\mathbf{L}$ is always **symmetric and positive definite**:

$$\mathbf{L} = \mathbf{L}^T > 0 \qquad \Rightarrow \qquad \mathbf{L} = \begin{bmatrix} L_a & 0 & 0 \\ 0 & J_m & 0 \\ 0 & 0 & C_0 \end{bmatrix}$$

2) The ***energy*** E_s ***stored in the system*** can be expressed as follows:

$$E_s = \frac{1}{2}\mathbf{x}^T \mathbf{L}\,\mathbf{x} \geq 0 \qquad \Rightarrow \qquad E_s = \frac{1}{2}L_a I_a^2 + \frac{1}{2}J_m \omega_m^2 + \frac{1}{2}C_0 P_0^2,$$

Matrix $\mathbf{L}$ is characterized by the **coefficients of the constitutive relations** of the dynamic elements of the system.

3) The ***power*** P_d ***dissipated in the system*** can be expressed as follows:

$$P_d = \mathbf{x}^T \mathbf{A}_s\,\mathbf{x} \qquad \Rightarrow \qquad P_d = -R_a I_a^2 - b_m \omega_m^2 - \alpha_p P_0^2$$

(if $\mathbf{L}$ is constant) where $\mathbf{A}_s$ is the symmetric part of the power matrix $\mathbf{A}$.

$$\mathbf{A}_s = \frac{(\mathbf{A} + \mathbf{A}^T)}{2} = \begin{bmatrix} -R_a & 0 & 0 \\ 0 & -b_m & 0 \\ 0 & 0 & -\alpha_p \end{bmatrix}$$

Matrix $\mathbf{A}_s$ is characterized by all the **dissipative** coefficients of the static elements present in the system.

4) The ***power*** P_w ***redistributed within the system*** is zero:

$$P_w = \mathbf{x}^T \mathbf{A}_w\,\mathbf{x} = 0$$

where $\mathbf{A}_w$ is the skew-symmetric part the power matrix $\mathbf{A}$:

$$\mathbf{A}_w = \frac{(\mathbf{A} - \mathbf{A}^T)}{2} = \begin{bmatrix} 0 & -K_m & 0 \\ K_m & 0 & -K_p \\ 0 & K_p & 0 \end{bmatrix}$$

Matrix $\mathbf{A}_w$ is characterized by all the coefficients of the **connecting blocks** present in the system.

- ***Definition***. *A* **"Linear Time-invariant Power-Oriented Graph dynamic system" S** *is characterized by a state space differential equation having the following structure:*

$$\mathbf{S} = \begin{cases} \mathbf{L}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

where $\mathbf{L}$ *is the* **energy matrix**, $\mathbf{A}$ *is the* **power matrix**, $\mathbf{B}$ *is the* **input power matrix**, $\mathbf{C}$ *is the* **output matrix** *and* $\mathbf{D}$ *is the* **input-output matrix**. *Moreover, a POG dynamic system satisfies the following properties:*

a) matrix $\mathbf{L} = \mathbf{L}^T \geq 0$ *is a symmetric semidefinite matrix;*

b) the energy E_s *stored in the system can be expressed as follows:*

$$E_s = \frac{1}{2}\mathbf{x}^T\mathbf{L}\mathbf{x} \geq 0$$

c) the power P_d *dissipated in the system can be expressed as follows:*

$$P_d = \mathbf{x}^T\mathbf{A}\mathbf{x}.$$

- ***Transformed POG systems***. A POG dynamic system $\mathbf{S}$ can be transformed using a ***"congruent"*** transformation $\mathbf{x} = \mathbf{T}\,\mathbf{z}$:

$$\mathbf{S} = \begin{cases} \mathbf{L}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u} \end{cases} \quad \overset{\mathbf{x}=\mathbf{T}\,\mathbf{z}}{\Longrightarrow} \quad \overline{\mathbf{S}} = \begin{cases} \overline{\mathbf{L}}\,\dot{\mathbf{z}} = \overline{\mathbf{A}}\,\mathbf{z} + \overline{\mathbf{B}}\,\mathbf{u} \\ \mathbf{y} = \overline{\mathbf{C}}\,\mathbf{z} + \mathbf{D}\,\mathbf{u} \end{cases}$$

where $\overline{\mathbf{S}}$ is the transformed system and (if matrix $\mathbf{T}$ is constant)

$$\overline{\mathbf{L}} = \mathbf{T}^T\mathbf{L}\mathbf{T}, \qquad \overline{\mathbf{A}} = \mathbf{T}^T\mathbf{A}\mathbf{T}, \qquad \overline{\mathbf{B}} = \mathbf{T}^T\mathbf{B}, \qquad \overline{\mathbf{C}} = \mathbf{C}\mathbf{T}$$

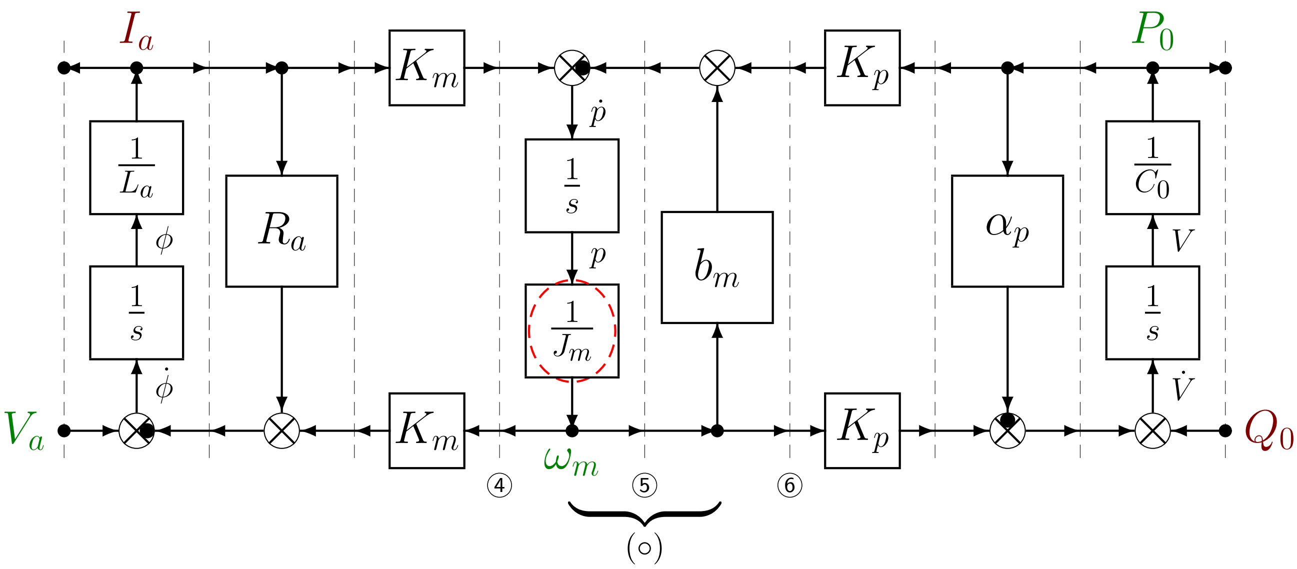
- The transformed POG system $\overline{\mathbf{S}}$ maintains the same properties of the original POG system $\mathbf{S}$.

- A **"congruent"** transformation $\mathbf{x} = \mathbf{T}\,\mathbf{z}$ does not require the calculation of the the inverse of matrix $\mathbf{T}$. It can be applied also when $\mathbf{T}$ is "singular" or "rectangular".

Model reduction of a POG block scheme

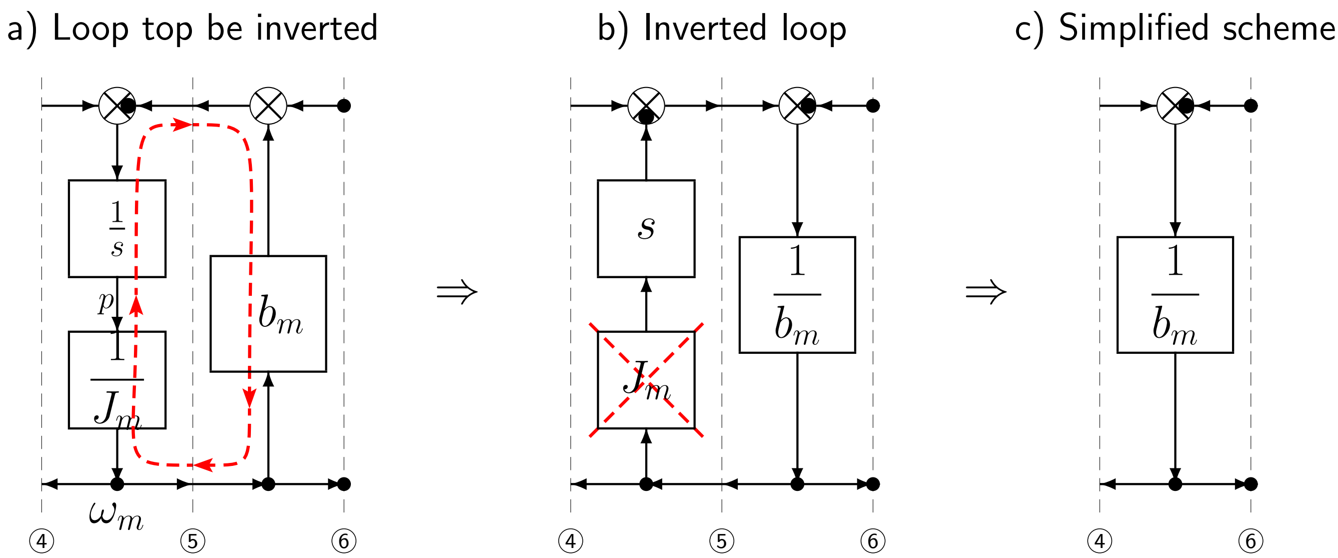
- When a physical parameter of the system tends to zero (or to infinity) the POG system degenerates towards a lower dimension dynamic system. The POG dynamic model can be reduced "**graphically**" or "**analytically**"

- **Graphical model reduction**. Let us consider the following POG scheme:

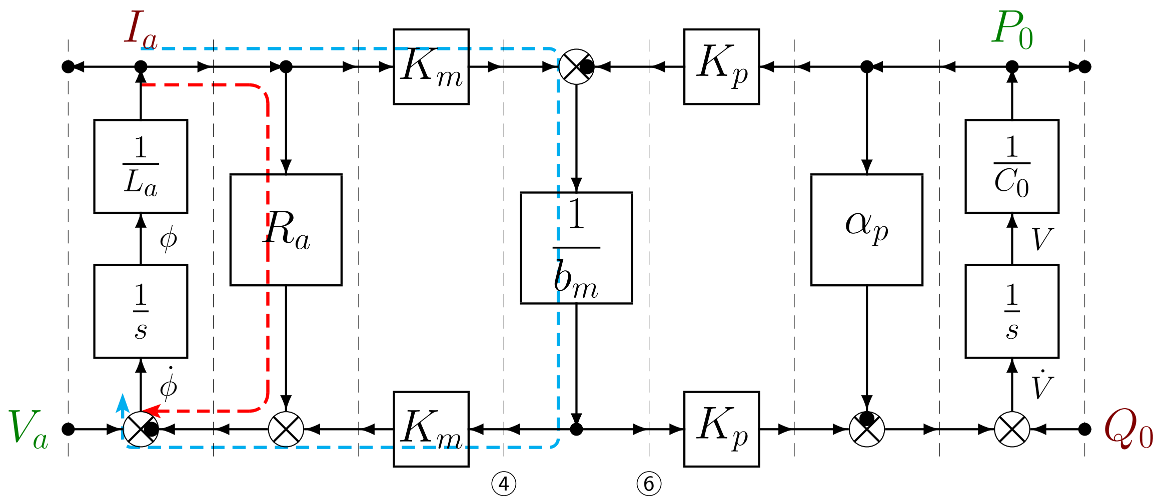


- When $J_m = 0$ the above POG block scheme cannot be used because the term $\frac{1}{J_m}$ present in the block scheme is infinite.

- In this case the central part (○) of the block scheme can be graphically transformed as follows:



- The simplified and transformed system has now the following structure:



- The corresponding POG state space model is:

$$\begin{bmatrix} L_a & 0 \\ 0 & C_0 \end{bmatrix} \begin{bmatrix} \dot{I}_a \\ \dot{P}_0 \end{bmatrix} = \begin{bmatrix} -R_a - \frac{K_m^2}{b_m} & \frac{K_m K_p}{b_m} \\ \frac{K_m K_p}{b_m} & -\alpha_p - \frac{K_p^2}{b_m} \end{bmatrix} \begin{bmatrix} I_a \\ P_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_a \\ Q_0 \end{bmatrix} \qquad (*)$$

- <u>**Analytical model reduction**</u>. When $J_m = 0$, the state space dynamic model of the system can be rewritten as follows:

$$\begin{bmatrix} L_a & 0 & 0 \\ 0 & \mathbf{0} & 0 \\ 0 & 0 & C_0 \end{bmatrix} \begin{bmatrix} \dot{I}_a \\ \dot{\omega}_m \\ \dot{P}_0 \end{bmatrix} = \begin{bmatrix} -R_a & -K_m & 0 \\ K_m & -b_m & -K_p \\ 0 & K_p & -\alpha_p \end{bmatrix} \begin{bmatrix} I_a \\ \omega_m \\ P_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_a \\ Q_0 \end{bmatrix}$$

The second equation is an algebraic constraint between the state variables:

$$K_m I_a - b_m \omega_m - K_p P_0 = 0$$

The angular velocity ω_m can be expressed as follows:

$$\omega_m = \frac{K_m}{b_m} I_a - \frac{K_p}{b_m} P_0$$

- Applying the following "congruent" state space transformation:

$$\underbrace{\begin{bmatrix} I_a \\ \omega_m \\ P_0 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{K_m}{b_m} & -\frac{K_p}{b_m} \\ 0 & 1 \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} I_a \\ P_0 \end{bmatrix}}_{\mathbf{z}}, \qquad \leftrightarrow \qquad \mathbf{x} = \mathbf{T}\,\mathbf{z}$$

to the given system one directly obtains the reduced system (*).