Observability and Constructability

- *Observability problem*: compute the *initial state* $\mathbf{x}(t_0)$ using the information associated to the input and output functions $\mathbf{u}(t)$ and $\mathbf{y}(t)$ of the given dynamic system in the time interval $t \in [t_0, t_1]$.
  - *Definition*. The initial state $\mathbf{x}(t_0)$ of a dynamic system is compatible with the input and output functions $\mathbf{u}(\cdot)$ and $\mathbf{y}(\cdot)$ in the time interval $[t_0, t_1]$ if relation $\mathbf{y}(\tau) = \eta(\tau, t_0, \mathbf{x}(t_0), \mathbf{u}(\cdot))$ holds for $\tau \in [t_0, t_1]$.
  - Let:
$$\mathcal{E}^-(t_0, t_1, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$$
denote the *set of all the initial states* $\mathbf{x}(t_0)$ *compatible* with the input and output functions $\mathbf{u}(\cdot)$ and $\mathbf{y}(\cdot)$in the time interval $[t_0, t_1]$.

- *Constructability problem*: compute the *final state* $\mathbf{x}(t_1)$ using the information associated to the input and output functions $\mathbf{u}(t)$ and $\mathbf{y}(t)$ of the given dynamic system in the time interval $t \in [t_0, t_1]$.
  - *Definition*. A final state $\mathbf{x}(t_1)$ of a dynamic system is compatible with the input and output functions $\mathbf{u}(\cdot)$ and $\mathbf{y}(\cdot)$ in the time interval $[t_0, t_1]$ if it exists an initial state $\mathbf{x}(t_0) \in \mathcal{E}^-(t_0, t_1, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$ such $\mathbf{x}(t_1) = \psi(t_0, t_1, \mathbf{x}(t_0), \mathbf{u}(\cdot))$.
  - Let:
$$\mathcal{E}^+(t_0, t_1, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$$
denote the *set of all the final states* $\mathbf{x}(t_1)$ *compatible* with the input and output functions $\mathbf{u}(\cdot)$ and $\mathbf{y}(\cdot)$in the time interval $[t_0, t_1]$.

- For time-invariant systems the two sets $\mathcal{E}^-$ and $\mathcal{E}^+$ are function only of the amplitude $t_1 - t_0$ of the time interval and therefore in this case one can set $t_0 = 0$ and use the following simplified notation:
$$\mathcal{E}^-(t_1, \mathbf{u}(\cdot), \mathbf{y}(\cdot)), \qquad \mathcal{E}^+(t_1, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$$

Discrete-time linear systems

Let us consider the following discrete-time linear system $\mathcal{S} = (\mathbf{A},\, \mathbf{B},\, \mathbf{C},\, \mathbf{D})$:

$$\begin{cases} \mathbf{x}(k+1) &= \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\,\mathbf{x}(k) + \mathbf{D}\,\mathbf{u}(k) \end{cases}$$

Observability:

- *Definition*. The initial state $\mathbf{x}(0)$ is called unobservable in k steps if it belongs to the set $\mathcal{E}^-(k) = \mathcal{E}^-(k,\ 0,\ 0)$, that is if it is compatible with the zero input and zero output functions, $\mathbf{u}(\tau) = 0$ and $\mathbf{y}(\tau) = 0$, in the time interval $\tau \in [0,\ k-1]$:

$$0 = \mathbf{y}(\tau) = \mathbf{C}\mathbf{A}^\tau \mathbf{x}(0), \qquad \text{for } \tau \in [0,\ k-1]$$

that is if:

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(k-1) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{k-1} \end{bmatrix}}_{\mathcal{O}^-(k)} \mathbf{x}(0) = \mathcal{O}^-(k)\,\mathbf{x}(0)$$

where $\mathcal{O}^-(k)$ denotes the *observability matrix in k steps*.

- The set $\mathcal{E}^-(k)$ of all the initial states $\mathbf{x}(0)$ which are *unobservable in k steps* is a vectorial space which is equal to the kernel of matrix $\mathcal{O}^-(k)$:

$$\boxed{\mathcal{E}^-(k) = \ker[\mathcal{O}^-(k)]}$$

- The unobservable subspaces $\mathcal{E}^-(k)$ satisfy the following chain of inclusions (n is the dimension of the state space):

$$\mathcal{E}^-(1) \supseteq \mathcal{E}^-(2) \supseteq \ldots\ldots \supseteq \mathcal{E}^-(n) = \mathcal{E}^-(n+1) = \ldots$$

- The smallest unobservable subspace $\mathcal{E}^-(n)$ is obtained, at most, in n steps.

- The set $\mathcal{E}^-$ of all the initial states $\mathbf{x}(0)$ *unobservable* (*that is unobservable in any number of steps*) is a vectorial state space which is equal to the set $\mathcal{E}^-(n)$ of all the state space unobservable in n steps:

$$\boxed{\mathcal{E}^- = \mathcal{E}^-(n) = \ker[\mathcal{O}^-(n)]}$$

- The subspace $\mathcal{E}^-$ is called *unobservable subspace* of the system and can be determined as follows:

$$\boxed{\mathcal{E}^- = \ker\, \mathcal{O}^-} \qquad \mathcal{O}^- \triangleq \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

where $\mathcal{O}^- = \mathcal{O}^-(n)$ denotes the *observability matrix* of the system.

- *Definition*. A system $\mathcal{S}$ is called *observable (or completely observable)* if the subspace $\mathcal{E}^-$ is equal to the zero state 0.

- *Property* . A system $\mathcal{S}$ is *observable* if and only if one of these relations holds:

  - $\ker \mathcal{O}^- = \{0\}$
  - rango $\mathcal{O}^- = n$

- The initial state $\mathbf{x}(0) \in \mathcal{E}^-(k, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$ is compatible with the input and output functions $\mathbf{u}(\cdot)$ and $\mathbf{y}(\cdot)$ in the time interval $[0, k[$ if the following relation is satisfied for $\tau \in [0, k-1]$:

$$\mathbf{y}(\tau) = \mathbf{CA}^\tau \mathbf{x}(0) + \sum_{i=0}^{\tau-1} \mathbf{CA}^{\tau-1-i}\mathbf{Bu}(i) + \mathbf{D}\,\mathbf{u}(\tau)$$

that is if

$$\boxed{\mathbf{y}_l(\tau) = \mathbf{CA}^\tau \mathbf{x}(0)}$$

where $\mathbf{y}_l(\tau) = \mathbf{y}(\tau) - \sum_{i=0}^{\tau-1} \mathbf{CA}^{\tau-1-i}\mathbf{Bu}(i) - \mathbf{D}\,\mathbf{u}(\tau)$ is the "**free output evolution**" of the system. Function $\mathbf{y}_l(\tau)$ can be easily computed if the input and output functions $\mathbf{u}(\tau)$ and $\mathbf{y}(\tau)$ are known for $\tau \in [0, k-1]$.

- For time-invariant linear system the following relation holds:

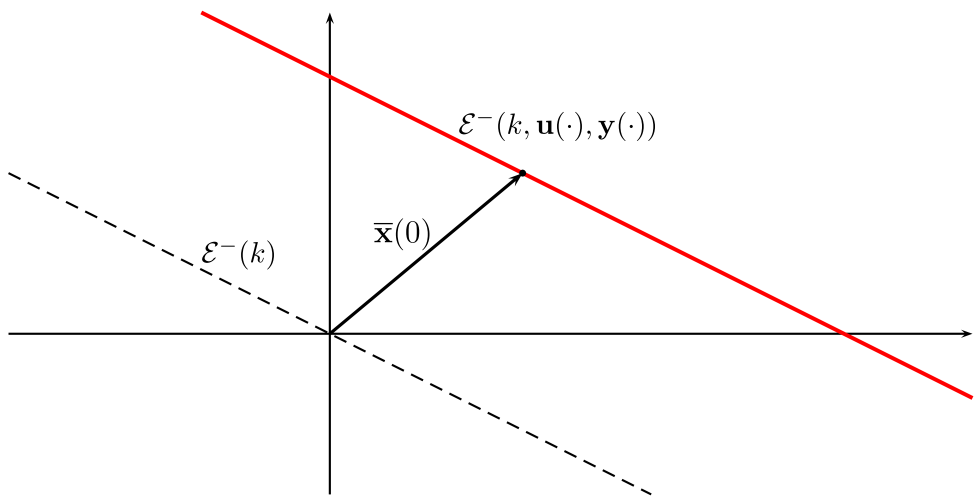
$$\boxed{\mathcal{E}^{-}(k, \mathbf{u}(\cdot), \mathbf{y}(\cdot)) = \mathcal{E}^{-}(k, 0, \mathbf{y}_l(\cdot))}$$

Note: the set $\mathcal{E}^{-}(k, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$ depends only on the parameter k and the fee output evolution $\mathbf{y}_l(\tau)$.

- *Property*. The set $\mathcal{E}^{-}(k, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$ is a "linear variety" which can be obtained adding a particular solution $\overline{\mathbf{x}}(0) \in \mathcal{E}^{-}(k, 0, \mathbf{y}_l(\cdot))$ to the unobservable subspace $\mathcal{E}^{-}(k) = \mathcal{E}^{-}(k, 0, 0)$ in k steps.

$$\boxed{\mathcal{E}^{-}(k, \mathbf{u}(\cdot), \mathbf{y}(\cdot)) = \overline{\mathbf{x}}(0) + \mathcal{E}^{-}(k)}$$

- Graphical representation:



- Setting $k = n$ in the previous relation, one obtains the set $\mathcal{E}^{-}(\mathbf{u}(\cdot), \mathbf{y}(\cdot))$ of all the initial states $\mathbf{x}(0)$ which are compatible with the functions $\mathbf{u}(\cdot)$ and $\mathbf{y}(\cdot)$ in any number of steps:

$$\boxed{\mathcal{E}^{-}(\mathbf{u}(\cdot), \mathbf{y}(\cdot)) = \overline{\mathbf{x}}(0) + \mathcal{E}^{-}}$$

This set contains only one element $\overline{\mathbf{x}}(0)$ for each couple of functions $\mathbf{u}(\cdot)$ and $\mathbf{y}(\cdot)$ *is and only if* $\mathcal{E}^{-} = 0$, that is if the system is completely observable. So, the following property holds.

- *Property*. If a linear system is observable, then it exists *only one initial state* $\overline{\mathbf{x}}(0)$ compatible with any couple of input and output functions $\mathbf{u}(k)$ and $\mathbf{y}(k)$ in the time inteval $k \in [0, n]$.

Constructability:

- Is the problem of finding the final state $\mathbf{x}(k)$ at time k compatible with the input and output functions $\mathbf{u}(0), \ldots, \mathbf{u}(k-1)$ and $\mathbf{y}(0), \ldots, \mathbf{y}(k-1)$.
- If the system is observable in k steps, then the set $\mathcal{E}^-(k, \mathbf{u}(\cdot), \mathbf{y}(\cdot))$ contains only one element $\overline{\mathbf{x}}(0)$ and therefore the final state $\overline{\mathbf{x}}(k)$ is unique:

$$\overline{\mathbf{x}}(k) = \mathbf{A}^k \overline{\mathbf{x}}(0) + \sum_{i=1}^{k-1} \mathbf{A}^{k-1-i} \mathbf{B} \mathbf{u}(i)$$

- On the contrary, if the system is unobservable in k steps, then it is $\mathcal{E}^-(k, \mathbf{u}(\cdot), \mathbf{y}(\cdot)) = \overline{\mathbf{x}}(0) + \mathcal{E}^-(k)$, and therefore the set $\mathcal{E}^+(k,\, \mathbf{u}(\cdot),\, \mathbf{y}(\cdot))$, of all the final states $\overline{\mathbf{x}}(k)$ compatible with the input and output functions $\mathbf{u}(\cdot)$ and $\mathbf{y}(\cdot)$ has the following structure:

$$\mathcal{E}^+(k,\, \mathbf{u}(\cdot),\, \mathbf{y}(\cdot)) = \overline{\mathbf{x}}(k) + \underbrace{\mathbf{A}^k \mathcal{E}^-(k)}_{\mathcal{E}^+(k)}$$

- The set $\mathcal{E}^+(k,\, \mathbf{u}(\cdot),\, \mathbf{y}(\cdot))$ is obtained adding a particular solution $\overline{\mathbf{x}}(k)$ to the subspace $\mathcal{E}^+(k)$ of all the final states "non constructable" in k steps:

$$\boxed{\mathcal{E}^+(k) = \mathbf{A}^k \mathcal{E}^-(k)}$$

- A system is *constructable in k steps* if $\mathcal{E}^+(k) = \mathbf{A}^k \mathcal{E}^-(k) = \{0\}$, that is

$$\boxed{\mathcal{E}^-(k) \subseteq \ker \mathbf{A}^k}$$

- A system is *constructable* if it exists a k such that $\mathcal{E}^-(k) \subseteq \ker \mathbf{A}^k$.
- It takes n steps, at most, for a discrete system to be constructable. So, the constructability condition can be expressed in the following way:

$$\boxed{\mathcal{E}^- = \ker \mathcal{O}^- \subseteq \ker \mathbf{A}^n}$$

- *Note*: from this last relation it follows that the observabilty implies the constructability, but not vice versa:

Observability $\Rightarrow$ Constructability

Example. Let us consider the following discrete linear system:

$$\begin{cases} \mathbf{x}(k+1) &= \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \mathbf{x}(k) \end{cases}$$

Compute the set of all the initial states $\mathbf{x}_0 = \mathbf{x}(0)$ compatible with the *free output evolution*: $y(0) = 2$, $y(1) = 2$, $y(2) = 0$.

The observability matrix of the system is:

$$\mathcal{O}^- = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & -2 & 2 \end{bmatrix}, \qquad \det \mathcal{O}^- = 4$$

Since $\mathcal{O}^-$ is not singular, the system is completely observable, and therefore if the system has a solution, then it has "only one" solution. The solution $\mathbf{x}_0$ can be determined as follows:

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} \mathbf{x}_0 = \mathcal{O}^- \mathbf{x}_0 \qquad \rightarrow \qquad \mathbf{x}_0 = [\mathcal{O}^-]^{-1} \begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix}$$

that is:

$$\mathbf{x}_0 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

So, the only initial state compatible with the given free output evolution is $\mathbf{x}_0 = [0,\ 1,\ 1]$.

<u>Note</u>: the free output evolution $y(\tau)$ is also determined for $\tau \geq 3$:

$$y(\tau) = \mathbf{CA}^\tau \mathbf{x}_0$$

Example. Let us consider the following discrete linear system:

$$\begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{x}(k) \end{cases}$$

Compute the set of all the initial states $\mathbf{x}_0 = \mathbf{x}(0)$ compatible with the free output evolution: $y(0) = 2$, $y(1) = 2$, $y(2) = 4$.

The system is not completely observable:

$$\mathcal{O}^- = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 2 & 2 \\ -4 & 4 & 4 \end{bmatrix} \qquad \rightarrow \qquad \mathcal{E}^- = \text{span} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The set of all the initial conditions compatible with the free output evolution $y(0) = 2$, $y(1) = 2$, $y(2) = 4$ can be determined computing a particular solution $\mathbf{x}_p$ and adding the unobservable space $\mathcal{E}^-$. The particular solution $\mathbf{x}_p$ can be determined solving the following equation

$$\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 2 & 2 \\ -4 & 4 & 4 \end{bmatrix} \mathbf{x}_p \qquad \leftrightarrow \qquad \mathbf{y} = \mathcal{O}^- \mathbf{x}_p$$

A solution exists because the vector $\mathbf{y}$ belongs to the image of matrix $\mathcal{O}^-$.

$$\mathbf{y} \in \text{Im}(\mathcal{O}^-) \qquad \rightarrow \qquad \mathbf{x}_p = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

So, the set of all the initial states $\mathbf{x}_0$ compatible with the free output evolution is the following:

$$\mathbf{x}_0 = \mathbf{x}_p + \mathcal{E}^- = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + \text{span} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Example. Let us consider the following discrete linear system:

$$\begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \mathbf{x}(k) \end{cases}$$

Compute the set of all the initial states $\mathbf{x}_0 = \mathbf{x}(0)$ compatible with the free output evolution:: $y(0) = 1$, $y(1) = 1$, $y(2) = 1$.

The observability matrix of the system is:

$$\mathcal{O}^- = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 4 & 2 & 0 \end{bmatrix}$$

Since the rank of matrix $\mathcal{O}^-$ is 2, the system is not completely observable. The set of all the initial states $\mathbf{x}_0 = \mathbf{x}(0)$ compatible with the free output evolution: $y(0) = 1$, $y(1) = 1$, $y(2) = 1$ satisfies the following relation:

$$\mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix} \neq \mathcal{O}^- \mathbf{x}_0 \qquad \leftrightarrow \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 4 & 2 & 0 \end{bmatrix} \mathbf{x}_0$$

The vector $\mathbf{y}$ cannot be obtained as linear combination of the columns of matrix $\mathcal{O}^-$, and therefore the given problem does not have solutions. There are no initial states $\mathbf{x}_0$ which are compatible with the given free output evolution.

Property. The initial state $\overline{\mathbf{x}}_0$ of minimum norm which minimizes the Euclidean norm $||\mathbf{y} - \mathcal{O}^- \mathbf{x}_0||$ is obtained as follows:

$$\overline{\mathbf{x}}_0 = (\mathcal{O}^-)^T \mathbf{N} (\mathbf{N}^T \mathcal{O}^- (\mathcal{O}^-)^T \mathbf{N})^{-1} \mathbf{N}^T \mathbf{y}$$

where matrix $\mathbf{N}$ is a base matrix of $\mathrm{Im}(\mathcal{O}^-)$:

$$\mathrm{Im}(\mathbf{N}) = \mathrm{Im}(\mathcal{O}^-)$$

The columns of matrix $\mathbf{N}$ are a set of linearly independent columns of matrix $\mathcal{O}^-$.

Invariant continuous-time linear systems

Let $\mathcal{S} = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be an invariant continuous-time linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{cases}$$

The observability and constructability properties of the continuous-time systems are very similar to those described for of the discrete-time system.

Observability:

- Definition. An initial state $\mathbf{x}(0)$ is unobservable in the time interval $[0,\ t]$ if it is compatible with the zero input and output functions, $\mathbf{u}(\tau) = 0$ and $\mathbf{y}(\tau) = 0$, for $\tau \in [0,\ t]$:

$$0 = \mathbf{C}\,e^{\mathbf{A}\,\tau}\mathbf{x}(0), \qquad \text{for } \tau \in [0,\ t]$$

- *The set $\mathcal{E}^-(t)$ of the initial states $\mathbf{x}(0)$ unobservable in $[0,\ t]$ is a vectorial subspace:*

$$\boxed{\mathcal{E}^-(t) = \ker O_t}$$

where O_t denotes the linear operator $O_t : \mathbf{X} \to \mathcal{Y}(t)$ which provides the free output evolution $y(\tau) \in \mathcal{Y}(t)$ corresponding to the initial state $\mathbf{x}(0) \in \mathbf{X}$:

$$O_t : \mathbf{x}(0) \to \mathbf{C}e^{\mathbf{A}\tau}\mathbf{x}(0), \qquad 0 \le \tau \le t$$

- Property. The unobservable subspace $\mathcal{E}^- = \mathcal{E}^-(t)$ does not depend on the length of the time interval $t > 0$ and it is equal to the kernel of the same *observability matrix* $\mathcal{O}^-$ defied for the discrete-time systems.

$$\boxed{\mathcal{E}^- = \ker[\mathcal{O}^-]}$$

- *So, for continuous-time systems, the computation of the initial state $\mathbf{x}(0)$ compatible with the input and output functions $\mathbf{u}(\cdot)$ and $\mathbf{y}(\cdot)$ does not depend on the length of he time interval $[0,\ t]$, provided it is not zero, but it depends only on the rank of the observability matrix $\mathcal{O}^-$.*

Constructability:

- For continuous-time system the *observability* is equivalent to the *constructability*. In fact, in this case, the unobservable and the unconstructable subspaces $\mathcal{E}^-$ and $\mathcal{E}^+$ are linked by the following relation:

$$\boxed{\mathcal{E}^+ = e^{\mathbf{A}t}\,\mathcal{E}^-}$$

- Since matrix $e^{\mathbf{A}t}$ is always invertible, for any matrix $\mathbf{A}$, the two subspaces $\mathcal{E}^+$ and $\mathcal{E}^-$ always have the same dimension. Moreover, it can be proved that they are equal:

$$\boxed{\mathcal{E}^+ = \mathcal{E}^-}$$

- From this last relation it follows that for continuous-time systems the osservability implies and is implied by the constructability:

$$\boxed{\text{Observability} \Leftrightarrow \text{Constructability}}$$

Duality

- Given a continuous or discrete-time system $\mathcal{S} = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, the system $\mathcal{S}_D = (\mathbf{A}^T, \mathbf{C}^T, \mathbf{B}^T, \mathbf{D}^T)$, is called the *dual system* of system $\mathcal{S}$.

No. of inputs of system $\mathcal{S}$	$=$	No. of outputs of system $\mathcal{S}_D$
No. of outputs of system $\mathcal{S}$	$=$	No. of inputs of system $\mathcal{S}_D$

- *Reachability and observability properties*. The reachability and observability matrices $\mathcal{R}_D^+$ and $\mathcal{O}_D^-$ of the dual system $\mathcal{S}_D$ are linked to the reachability and observability matrices $\mathcal{R}^+$ and $\mathcal{O}^-$ of system $\mathcal{S}$ as follows:

$$\mathcal{R}_D^+ = [\mathbf{C}^T \; \mathbf{A}^T\mathbf{C}^T \ldots (\mathbf{A}^T)^{n-1}\mathbf{C}^T] = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}^T = (\mathcal{O}^-)^T$$

$$\mathcal{O}_D^- = \begin{bmatrix} \mathbf{B}^T \\ \mathbf{B}^T\mathbf{A}^T \\ \vdots \\ \mathbf{B}^T(\mathbf{A}^T)^{n-1} \end{bmatrix} = [\mathbf{B} \; \mathbf{AB} \ldots \mathbf{A}^{n-1}\mathbf{B}]^T = (\mathcal{R}^+)^T$$

- *Property*. For the systems $\mathcal{S}$ and $\mathcal{S}_D$ the following properties hold:

$\mathcal{S}$ reachable	$\Leftrightarrow$	$\mathcal{S}_D$ observable
$\mathcal{S}$ observable	$\Leftrightarrow$	$\mathcal{S}_D$ reachable
$\mathcal{S}$ controllable	$\Leftrightarrow$	$\mathcal{S}_D$ constructable
$\mathcal{S}$ constructable	$\Leftrightarrow$	$\mathcal{S}_D$ controllable

- If two equivalent systems $\mathcal{S}$ and $\mathcal{S}'$ are linked by the state space transformation $\mathbf{x} = \mathbf{T}\,\mathbf{x}'$, then the corresponding dual system $\mathcal{S}_D$ and $\mathcal{S}'_D$ are linked by the state space transformation $\mathbf{x}_D = \mathbf{P}\,\mathbf{x}'_D$ where $\boxed{\mathbf{P} = \mathbf{T}^{-T}}$:

$$\mathcal{S} = (\mathbf{A},\ \mathbf{B},\ \mathbf{C}) \quad \overset{\mathbf{T}}{\Rightarrow} \quad \mathcal{S}' = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T},\ \mathbf{T}^{-1}\mathbf{B},\ \mathbf{C}\mathbf{T})$$

$\Updownarrow$ Duality $\qquad\qquad$ $\Updownarrow$ Duality

$$\mathcal{S}_D = (\mathbf{A}^T,\ \mathbf{C}^T,\ \mathbf{B}^T) \quad \overset{\mathbf{P}}{\Rightarrow} \quad \mathcal{S}'_D = (\mathbf{T}^T\mathbf{A}^T\mathbf{T}^{-T},\ \mathbf{T}^T\mathbf{C}^T,\ \mathbf{B}^T\mathbf{T}^{-T})$$

Osservability standard form

- *<u>Property</u>. A linear system $\mathcal{S} = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ (continuous or discrete) <u>not</u> completely observable can always be brought into "observability standard form", that is, system $\mathcal{S}$ is algebraically equivalent the a transformed system $\overline{\mathcal{S}} = \{\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\mathbf{C}}, \overline{\mathbf{D}}\}$ where the matrices $\overline{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, $\overline{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B}$, $\overline{\mathbf{C}} = \mathbf{C}\mathbf{P}$ and $\overline{\mathbf{D}}$ have the following structure:*

$$\overline{\mathbf{A}} = \begin{bmatrix} \overline{\mathbf{A}}_{1,1} & \mathbf{0} \\ \overline{\mathbf{A}}_{2,1} & \overline{\mathbf{A}}_{2,2} \end{bmatrix} \qquad \overline{\mathbf{B}} = \begin{bmatrix} \overline{\mathbf{B}}_1 \\ \overline{\mathbf{B}}_2 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \overline{\mathbf{C}}_1 & \mathbf{0} \end{bmatrix} \qquad \overline{\mathbf{D}} = \mathbf{D}$$

- Let be $\rho = n - \text{dim}\mathcal{E}^- < n$. The transformation matrix $\mathbf{P}$ which brings the system into the observability standard form has the following structure:

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix}, \qquad \mathbf{x} = \mathbf{P}\,\overline{\mathbf{x}}$$

  where $\mathbf{R}_1 \in \mathbf{R}^{\rho\times n}$ is composed by ρ linearly independent rows of the observability matrix $\mathcal{O}^-$, and $\mathbf{R}_2 \in \mathbf{R}^{(n-\rho)\times n}$ is a free matrix chosen such that the transformation matrix $\mathbf{P}^{-1}$ is non singular.

- *<u>Proprerty</u>. The subsystem of dimension ρ characterized by matrices $\overline{\mathbf{A}}_{1,1}$ and $\overline{\mathbf{C}}_1$ is completely observable.*

- The subsystem $(\overline{\mathbf{A}}_{1,1},\ \overline{\mathbf{B}}_1,\ \overline{\mathbf{C}}_1)$, is called <u>*observable subsystem*</u> and it represents the part of the original system which is observable.

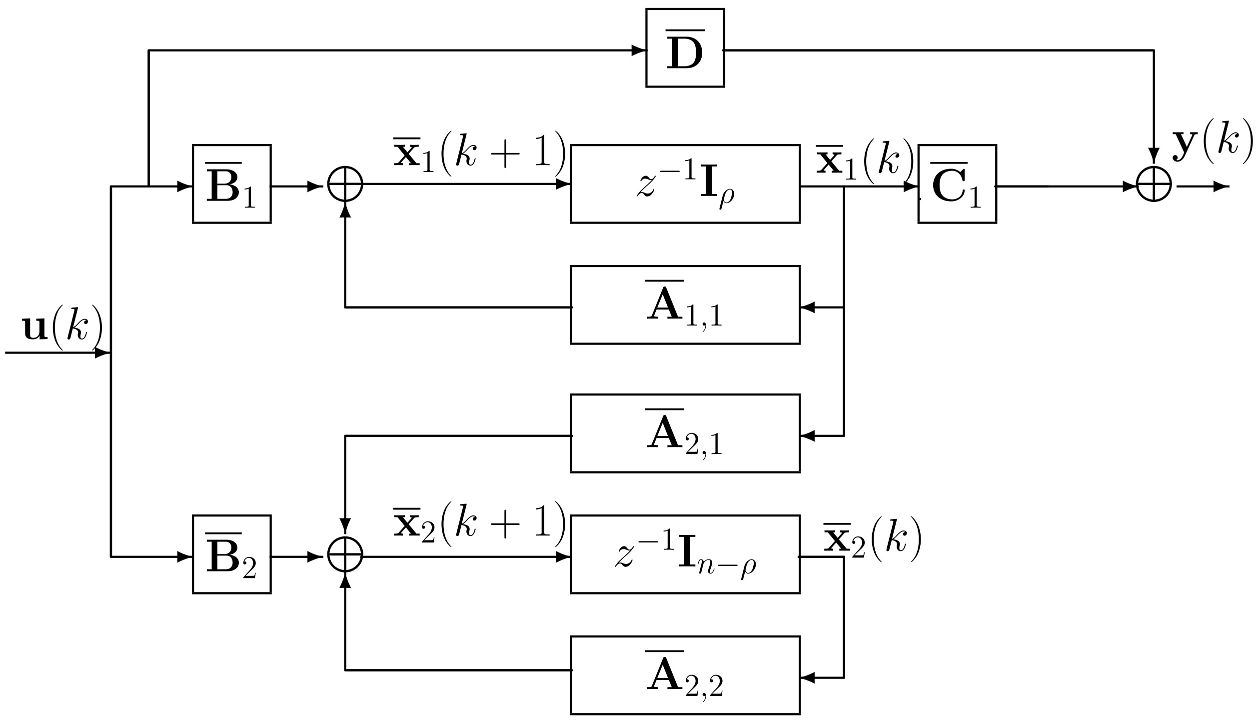
- The subsystem $(\overline{\mathbf{A}}_{2,2},\ \overline{\mathbf{B}}_2,\ 0)$, characterized by matrices $\overline{\mathbf{A}}_{2,2}$, $\overline{\mathbf{B}}_2$ and $\overline{\mathbf{C}}_2 = 0$, is called <u>*unobservable subsystem*</u> and it represents the part of the system which does not influence in any way the system output $\mathbf{y}$.

- Also in this case the eigenvalues of matrix $\mathbf{A}$ are split in two sets: the eigenvalues of the observable part of the system (the eigenvalues of matrix $\overline{\mathbf{A}}_{1,1}$) and the eigenvalues of the unobservable part (the eigenvalues of of matrix $\overline{\mathbf{A}}_{2,2}$).

- For the discrete case, let us divide the state vector $\overline{\mathbf{x}}(k)$ in two parts: the *observable* part $\overline{\mathbf{x}}_1$ and the *unobservable* part $\overline{\mathbf{x}}_2$: $\overline{\mathbf{x}} = \left[\begin{array}{cc} \overline{\mathbf{x}}_1 & \overline{\mathbf{x}}_2 \end{array}\right]^T$, where $dim\,\overline{\mathbf{x}}_1 = \rho$. The equations of the system are:

$$\begin{cases} \overline{\mathbf{x}}_1(k+1) = & \overline{\mathbf{A}}_{1,1}\overline{\mathbf{x}}_1(k) + & & \overline{\mathbf{B}}_1\mathbf{u}(k) \\ \overline{\mathbf{x}}_2(k+1) = & \overline{\mathbf{A}}_{2,1}\overline{\mathbf{x}}_1(k) + & \overline{\mathbf{A}}_{2,2}\overline{\mathbf{x}}_2(k) + & \overline{\mathbf{B}}_2\mathbf{u}(k) \\ \mathbf{y}(k) = & \overline{\mathbf{C}}_1\overline{\mathbf{x}}_1(k) + & & \overline{\mathbf{D}}\mathbf{u}(k) \end{cases}$$

The corresponding block scheme is:



- A similar decomposition holds also for continuous-time systems.
- Property. *The transfer matrix* $\mathbf{H}(z)$ *[o* $\mathbf{H}(s)$ *] of a linear system is equal to the transfer matrix of the* observable part*:* $\mathbf{H}(z)$ *[o* $\mathbf{H}(s)$ *] is influenced only by the matrices (*$\overline{\mathbf{A}}_{1,1}$, $\overline{\mathbf{B}}_1$, $\overline{\mathbf{C}}_1$*) of the observable subsystem.*

  Proof. The transfer matrix $\mathbf{H}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \overline{\mathbf{C}}(z\mathbf{I} - \overline{\mathbf{A}})^{-1}\overline{\mathbf{B}}$ is:

$$\begin{aligned} \mathbf{H}(z) &= \left[\begin{array}{cc} \mathbf{C}_1 & 0 \end{array}\right] \left[\begin{array}{cc} z\mathbf{I} - \mathbf{A}_{1,1} & 0 \\ -\mathbf{A}_{2,1} & z\mathbf{I} - \mathbf{A}_{2,2} \end{array}\right]^{-1} \left[\begin{array}{c} \mathbf{B}_1 \\ \mathbf{B}_2 \end{array}\right] = \\ &= \left[\begin{array}{cc} \mathbf{C}_1 & 0 \end{array}\right] \left[\begin{array}{cc} (z\mathbf{I} - \mathbf{A}_{1,1})^{-1} & 0 \\ **** & (z\mathbf{I} - \mathbf{A}_{2,2})^{-1} \end{array}\right] \left[\begin{array}{c} \mathbf{B}_1 \\ \mathbf{B}_2 \end{array}\right] = \\ &= \mathbf{C}_1(z\mathbf{I} - \mathbf{A}_{1,1})^{-1}\mathbf{B}_1 \end{aligned}$$

Example. Let us consider the following time-continuous linear system

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \mathbf{u}(t) \\ y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

Bring the system in the observability standard form.

Sol. The observability matrix of the system is:

$$\mathcal{O}^- = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & -2 \\ -3 & -3 & 0 \end{bmatrix} \qquad \rightarrow \qquad \det \mathcal{O}^- = 0$$

Matrix $\mathcal{O}^-$ is singular. The system is not completely observable and therefore it is possible to compute the transformation matrix $\mathbf{P}$ which brings the system into the observability standard form:

$$\mathbf{P}^{-1} = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array}\right] \qquad \rightarrow \qquad \mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

The transformed system has the following form:

$$\begin{cases} \dot{\bar{\mathbf{x}}}(t) = \left[\begin{array}{cc|c} 1 & -2 & \mathbf{0} \\ 2 & -1 & \mathbf{0} \\ \hline 1 & -1 & 0 \end{array}\right] \bar{\mathbf{x}}(t) + \left[\begin{array}{c} 2 \\ 0 \\ \hline 1 \end{array}\right] u(t) \\ y(t) = \left[\begin{array}{cc|c} 1 & 0 & \mathbf{0} \end{array}\right] \bar{\mathbf{x}}(t) \end{cases}$$

The unobservable part of the system is "simply stable" because it has an eigenvalues located in the origin: $s = 0$.

The transfer matrix:

$$G(s) = \mathbf{C}_o(s\mathbf{I} - \mathbf{A}_o)^{-1}\mathbf{B}_o = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 2 \\ -2 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{2(s+1)}{s^2+3}$$

if a function only of the observable part of the system.