Equilibrium points: continuous-time systems

- Let us consider the following continuous-time linear system

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{cases}$$

- The equilibrium points $\mathbf{x}_0$ of the system when the input is constant $\mathbf{u}(t) = \mathbf{u}_0$ can be determined imposing $\dot{\mathbf{x}} = 0$:

$$\mathbf{A}\,\mathbf{x}_0 + \mathbf{B}\,\mathbf{u}_0 = 0, \qquad \rightarrow \qquad \boxed{\mathbf{x}_0 = -\mathbf{A}^{-1}\,\mathbf{B}\,\mathbf{u}_0}$$

- If matrix $\mathbf{A}$ is invertible, the system has only one equilibrium point $\mathbf{x}_0$.
- If matrix $\mathbf{A}$ is singular (that is if matrix $\mathbf{A}$ has at least one eigenvalue in the origin) we can have two different cases:
  1) *there are infinite equilibrium points*. This situation happens when: $\mathrm{rank}[\mathbf{A}] = \mathrm{rank}[\mathbf{A}\,\mathbf{B}]$. In this case all the solutions can be determined adding the kernel of matrix $\mathbf{A}$ to a particular solution $\bar{\mathbf{x}}_0$:

$$\mathbf{x}_0 = \bar{\mathbf{x}}_0 + \ker[\mathbf{A}]$$

  2) *there are no equilibrium points*. This situation happens when: $\mathrm{rank}[\mathbf{A}] \neq \mathrm{rank}[\mathbf{A}\,\mathbf{B}]$.
- The output value $\mathbf{y}_0$ corresponding to the particular equilibrium point ($\mathbf{x}_0$, $\mathbf{u}_0$) can be directly obtained using the output equation:

$$\mathbf{y}_0 = \mathbf{C}\,\mathbf{x}_0 + \mathbf{D}\,\mathbf{u}_0$$

  If matrix $\mathbf{A}$ is invertible, the following relation holds:

$$\mathbf{y}_0 = -\mathbf{C}\,\mathbf{A}^{-1}\,\mathbf{B}\,\mathbf{u}_0 + \mathbf{D}\,\mathbf{u}_0 = [\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]_{s=0}\mathbf{u}_0 = \mathbf{H}(s)_{s=0}\mathbf{u}_0$$

  where $\mathbf{H}(s)$ denotes the transfer matrix of the system.
- For linear systems the stability of an equilibrium point does not depend on the point itself, but it depends on the stability of the system which is completely determined by the position of the eigenvalues of matrix $\mathbf{A}$.

Equilibrium points: discrete-time systems

- Let us consider the following discrete-time linear system:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\,\mathbf{x}(k) + \mathbf{D}\,\mathbf{u}(k) \end{cases}$$

- When the input is constant $\mathbf{u}(t) = \mathbf{u}_0$, the equilibrium points $\mathbf{x}_0$ of the system can be determined imposing $\mathbf{x}(k+1) = \mathbf{x}(k) = \mathbf{x}_0$:

$$\mathbf{x}_0 = \mathbf{A}\,\mathbf{x}_0 + \mathbf{B}\,\mathbf{u}_0, \qquad \rightarrow \qquad \boxed{\mathbf{x}_0 = (\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{u}_0}$$

- In this case the system has only one equilibrium point $\mathbf{x}_0$ if and only if matrix $(\mathbf{I} - \mathbf{A})$ is invertible.
- If matrix $(\mathbf{I} - \mathbf{A})$ is singular (that is if matrix $\mathbf{A}$ has at least one eigenvalue in $z = 1$), then the system:

  1) *has infinite equilibrium points* if $\mathrm{rank}[\mathbf{I} - \mathbf{A}] = \mathrm{rank}[(\mathbf{I} - \mathbf{A})\ \mathbf{B}]$:

  $$\mathbf{x}_0 = \bar{\mathbf{x}}_0 + \ker[\mathbf{I} - \mathbf{A}]$$

  2) *does not have equilibrium points* if $\mathrm{rank}[\mathbf{I} - \mathbf{A}] \neq \mathrm{rank}[(\mathbf{I} - \mathbf{A})\ \mathbf{B}]$.

- The output value $\mathbf{y}_0$ corresponding to the equilibrium point $(\mathbf{x}_0, \mathbf{u}_0)$ can be determined as follows:

$$\mathbf{y}_0 = \mathbf{C}\,\mathbf{x}_0 + \mathbf{D}\,\mathbf{u}_0.$$

- If matrix $(\mathbf{I} - \mathbf{A})$ is invertible, then the following relation holds:

$$\mathbf{y}_0 = \mathbf{C}\,(\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{u}_0 + \mathbf{D}\,\mathbf{u}_0 = [\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]_{z=1}\mathbf{u}_0 = \mathbf{H}(z)|_{z=1}\,\mathbf{u}_0$$

  where $\mathbf{H}(z)$ denotes the transfer matrix of the discrete system.

- The static gain $\mathbf{H}(z)_{z=1}$ of the transfer matrix $\mathbf{H}(z)$ is infinite if matrix $\mathbf{I} - \mathbf{A}$ has at least one eigenvalue in the origin, that is if matrix $\mathbf{A}$ has at least one eigenvalue in $z = 1$.

Equilibrium points: nonlinear systems

- Let us now consider the following continuous-time nonlinear system:

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}$$

When the input is constant $\mathbf{u}(t) = \mathbf{u}_0$, the equilibrium points $\mathbf{x}_0$ can be determined imposing $\dot{\mathbf{x}} = 0$:

$$\boxed{\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = 0}$$

- The nonlinear vectorial static equation $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = 0$ must be solved with respect to variable $\mathbf{x}_0$. For nonlinear systems all these cases are possible: 1) no equilibrium points; 2) only one equilibrium point; 3) a finite number of equilibrium points; 4) an infinite number of equilibrium points; ecc.

- The output value $\mathbf{y}_0$ corresponding to the equilibrium point ($\mathbf{x}_0$, $\mathbf{u}_0$) can be directly determined using the output equation:

$$\mathbf{y}_0 = \mathbf{g}(\mathbf{x}_0, \mathbf{u}_0).$$

- For nonlinear systems the stability is NOT a global property of the system, but a "local" property of the considered equilibrium point $\mathbf{x}_0$. In this case a stability analysis must be done for "each" equilibrium point $\mathbf{x}_0$.

- Let us now consider the case of a discrete-time nonlinear system:

$$\begin{cases} \mathbf{x}(k+1) &= \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) \\ \mathbf{y}(k) &= \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k)) \end{cases}$$

- When the input is constant $\mathbf{u}(t) = \mathbf{u}_0$, the equilibrium points $\mathbf{x}_0$ of the discrete system can be determined imposing $\mathbf{x}(k+1) = \mathbf{x}(k) = \mathbf{x}_0$:

$$\boxed{\mathbf{x}_0 = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)}$$

This nonlinear vectorial static equation must be solved with respect to unknown variable $\mathbf{x}_0$.

Linearization in the vicinity of an equilibrium pont

- Let us consider the following continuous-time nonlinear system:

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}$$

and let $\mathbf{x}_0$ be an equilibrium point of the system corresponding to the constant input $\mathbf{u}_0$.

- Expanding the functions $\mathbf{f}(\mathbf{x}, \mathbf{u})$ and $\mathbf{g}(\mathbf{x}, \mathbf{u})$ in the vicinity of the equilibrium point $(\mathbf{x}_0,\ \mathbf{u}_0)$ using the Taylor series, one obtains:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \underbrace{\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)}_{0} + \left.\frac{\partial \mathbf{f}(\mathbf{x},\ \mathbf{u})}{\partial\, \mathbf{x}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} (\mathbf{x}-\mathbf{x}_0) + \left.\frac{\partial \mathbf{f}(\mathbf{x},\ \mathbf{u})}{\partial\, \mathbf{u}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} (\mathbf{u}-\mathbf{u}_0) + \mathbf{h}_1(\mathbf{x}, \mathbf{u})$$

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) = \underbrace{\mathbf{g}(\mathbf{x}_0, \mathbf{u}_0)}_{\mathbf{y}_0} + \left.\frac{\partial \mathbf{g}(\mathbf{x},\ \mathbf{u})}{\partial\, \mathbf{x}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} (\mathbf{x}-\mathbf{x}_0) + \left.\frac{\partial \mathbf{g}(\mathbf{x},\ \mathbf{u})}{\partial\, \mathbf{u}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} (\mathbf{u}-\mathbf{u}_0) + \mathbf{h}_2(\mathbf{x}, \mathbf{u})$$

where $\mathbf{h}_1(\cdot)$ and $\mathbf{h}_2(\cdot)$ denote high order infinitesimals which are supposed to be negligible in the vicinity of the equilibrium point $(\mathbf{x}_0,\ \mathbf{u}_0)$.

- Using the new system variables $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$, $\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{y}_0$ and $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$, one obtains the following linearized system:

$$\begin{cases} \dot{\tilde{\mathbf{x}}}(t) &= \mathbf{A}\, \tilde{\mathbf{x}}(t) + \mathbf{B}\, \tilde{\mathbf{u}}(t) \\ \tilde{\mathbf{y}}(t) &= \mathbf{C}\, \tilde{\mathbf{x}}(t) + \mathbf{D}\, \tilde{\mathbf{u}}(t) \end{cases}$$

where the system matrices have the following structure:

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}(\mathbf{x},\ \mathbf{u})}{\partial\, \mathbf{x}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)}, \qquad \mathbf{B} = \left.\frac{\partial \mathbf{f}(\mathbf{x},\ \mathbf{u})}{\partial\, \mathbf{u}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)}$$

$$\mathbf{C} = \left.\frac{\partial \mathbf{g}(\mathbf{x},\ \mathbf{u})}{\partial\, \mathbf{x}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)}, \qquad \mathbf{D} = \left.\frac{\partial \mathbf{g}(\mathbf{x},\ \mathbf{u})}{\partial\, \mathbf{u}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)}$$

- The stability of a nonlinear system in the vicinity of an equilibrium point $(\mathbf{x}_0,\ \mathbf{u}_0)$ can be studied applying the "reduced Lyapunov criterium" to the linearized system.

- For discrete-time nonlinear system:

$$\left\{\begin{array}{rcl} \mathbf{x}(k+1) &=& \mathbf{f}(\mathbf{x}(k),\mathbf{u}(k)) \\ \mathbf{y}(k) &=& \mathbf{g}(\mathbf{x}(k),\mathbf{u}(k)) \end{array}\right.$$

  the linearization in the vicinity of an equilibrium point $(\mathbf{x}_0, \mathbf{u}_0)$ can be done exactly in the same way as it has been done for the continuous-time case:

$$\left\{\begin{array}{rcl} \tilde{\mathbf{x}}(k+1) &=& \mathbf{A}\,\tilde{\mathbf{x}}(k) + \mathbf{B}\,\tilde{\mathbf{u}}(k) \\ \tilde{\mathbf{y}}(k) &=& \mathbf{C}\,\tilde{\mathbf{x}}(k) + \mathbf{D}\,\tilde{\mathbf{u}}(k) \end{array}\right.$$

  The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ can be obtained using the same expressions shown above.

- For computing the system matrices, it is useful to remember that the nonlinear vectors $\mathbf{f}(\mathbf{x},\,\mathbf{u})$ e $\mathbf{g}(\mathbf{x},\,\mathbf{u})$ have the following structure:

$$\mathbf{f}(\mathbf{x},\,\mathbf{u}) = \begin{bmatrix} f_1(\mathbf{x},\,\mathbf{u}) \\ f_2(\mathbf{x},\,\mathbf{u}) \\ \vdots \\ f_n(\mathbf{x},\,\mathbf{u}) \end{bmatrix}, \qquad \mathbf{g}(\mathbf{x},\,\mathbf{u}) = \begin{bmatrix} g_1(\mathbf{x},\,\mathbf{u}) \\ g_2(\mathbf{x},\,\mathbf{u}) \\ \vdots \\ g_m(\mathbf{x},\,\mathbf{u}) \end{bmatrix}$$

  and that matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ can be computed as follows:

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u})}{\partial\,\mathbf{x}}\right|_{(\mathbf{x}_0,\mathbf{u}_0)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(\mathbf{x}_0,\mathbf{u}_0)}, \qquad \mathbf{B} = \left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u})}{\partial\,\mathbf{u}}\right|_{(\mathbf{x}_0,\mathbf{u}_0)} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{(\mathbf{x}_0,\mathbf{u}_0)}$$

$$\mathbf{C} = \left.\frac{\partial \mathbf{g}\mathbf{x},\mathbf{u})}{\partial\,\mathbf{x}}\right|_{(\mathbf{x}_0,\mathbf{u}_0)} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_p}{\partial x_1} & \cdots & \frac{\partial g_p}{\partial x_n} \end{bmatrix}_{(\mathbf{x}_0,\mathbf{u}_0)}, \qquad \mathbf{D} = \left.\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial\,\mathbf{u}}\right|_{(\mathbf{x}_0,\mathbf{u}_0)} = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial g_p}{\partial u_1} & \cdots & \frac{\partial g_p}{\partial u_m} \end{bmatrix}_{(\mathbf{x}_0,\mathbf{u}_0)}$$

- The obtained Jacobian matrices have the following dimensions: $\mathbf{A} \in \mathbf{R}^{n\times n}$, $\mathbf{B} \in \mathbf{R}^{n\times m}$, $\mathbf{C} \in \mathbf{R}^{p\times n}$ and $\mathbf{D} \in \mathbf{R}^{p\times m}$.