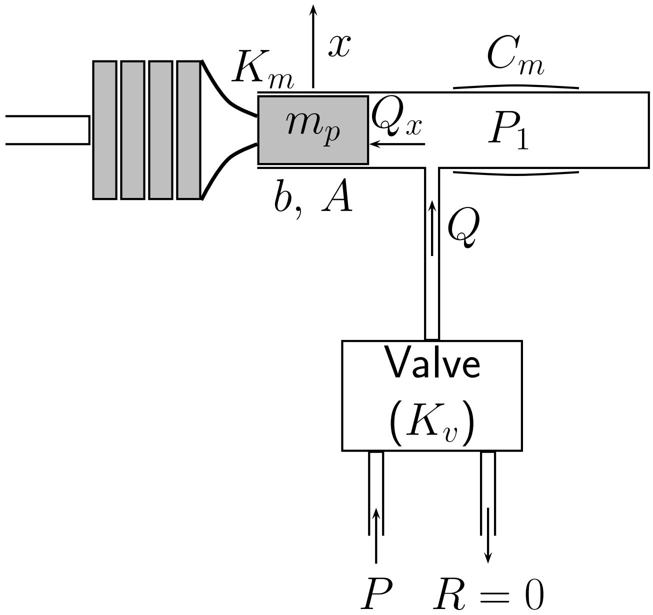
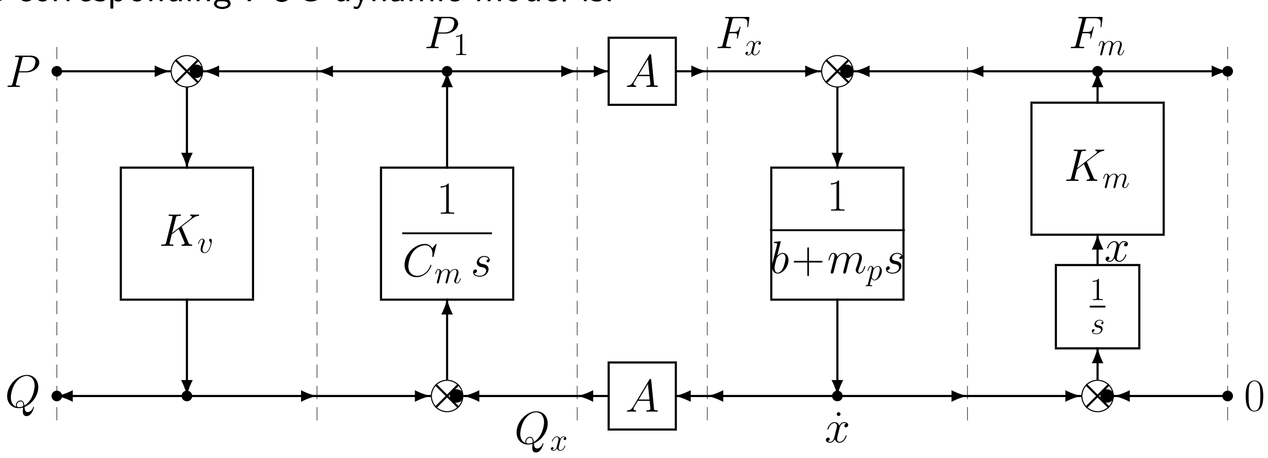
# Dynamic model of an hydraulic clutch

Let us consider the following simplified dynamic model of an hydraulic clutch:

| | |
|---|---|
| $P$ | Supply pressure |
| $Q$ | Flow rate through the valve |
| $K_v$ | Valve proportional constant |
| $C_m$ | Hydraulic capacity of the cylinder |
| $P_1$ | Pressure within the cylinder |
| $A$ | Section of the piston |
| $x$ | Position of the piston |
| $\dot{x}$ | Velocity of the piston |
| $m_p$ | Mass of the piston |
| $b$ | Friction coefficient of the piston |
| $K_m$ | Stiffness of the spring |
| $F_m$ | Spring force against the piston |




The corresponding POG dynamic model is:




Let us introduce the following auxiliary variables:

$$G_1 = K_v, \qquad G_2 = \frac{1}{C_m s}, \qquad G_3 = \frac{1}{b + m_p s}, \qquad G_4 = \frac{K_m}{s}$$

Using the Mason formula ones obtains the following transfer function:

$$G(s) = \frac{F_m(s)}{P(s)} = \frac{A\, G_1\, G_2\, G_3\, G_4}{1 + G_1\, G_2 + A^2\, G_2\, G_3 + G_3\, G_4 + G_1\, G_2\, G_3\, G_4}$$

which, by substitution, simplifies as follows:

$$G(s) = \frac{A K_m K_v}{C_m m_p s^3 + (C_m\, b + K_v m_p)s^2 + (A^2 + C_m K_m + K_v\, b)s + K_m K_v}$$

The corresponding differential equation is:

$$C_m m_p \dddot{F}_m + (C_m\, b + K_v m_p)\ddot{F}_m + (A^2 + C_m K_m + K_v\, b)\dot{F}_m + K_m K_v F_m = A K_m K_v P(t)$$

The system is characterized by the following static and differential equations:

$$\begin{cases} Q = K_v(P - P_1) \\ C_m\dot{P}_1 = Q - Q_x \\ m_p\ddot{x} = F_x - b\,\dot{x} - F_m \end{cases} \qquad \begin{cases} F_m = K_m\, x \\ F_x = A\, P_1 \\ Q_x = A\,\dot{x} \end{cases}$$

If the system equations are written without an order and without a method, it is difficult to get the dynamic order of the system or to understand if all the static and dynamic equations have been written. To cope with this problems it is useful to introduce a "state vector":

$$\mathbf{x} = \begin{bmatrix} P_1 & \dot{x} & x \end{bmatrix}^T$$

The system equations can now be rewritten as follows:

$$\begin{aligned} \begin{bmatrix} C_m\,\dot{P}_1 \\ m_p\,\ddot{x} \\ \dot{x} \end{bmatrix} &= \begin{bmatrix} -K_v & -A & 0 \\ A & -b & -K_m \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} K_v \\ 0 \\ 0 \end{bmatrix} P \\ \mathbf{y} &= F_m \end{aligned}$$

where $P$ is the input pressure and $\mathbf{y} = F_m$ is the output force. The system equations can also be rewritten as follows:

$$\begin{aligned} \underbrace{\begin{bmatrix} \dot{P}_1 \\ \ddot{x} \\ \dot{x} \end{bmatrix}}_{\dot{\mathbf{x}}} &= \underbrace{\begin{bmatrix} -\frac{K_v}{C_m} & -\frac{A}{C_m} & 0 \\ \frac{A}{m_p} & -\frac{b}{m_p} & -\frac{K_m}{m_p} \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} P_1 \\ \dot{x} \\ x \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \frac{K_v}{C_m} \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{P}_{\mathbf{u}} \\ \mathbf{y} &= \underbrace{\begin{bmatrix} 0 & 0 & K_m \end{bmatrix}}_{\mathbf{C}} \mathbf{x} \end{aligned}$$

which, in compact form, is described as follows:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{y} = \mathbf{C}\,\mathbf{x} \end{cases}$$

where
$\mathbf{A}$ is the system matrix
$\mathbf{B}$ is the input matrix
$\mathbf{C}$ is the output matrix

The system is linear and stable and therefore it has only one stable equilibrium point $\mathbf{x}_0$ for each constant input $\mathbf{u} = \mathbf{u}_0$.

The final equilibrium point $\mathbf{x}_0$ can be computed setting $\dot{\mathbf{x}} = 0$ and $\mathbf{u}_0 = P$:

$$\mathbf{x}_0 = -\mathbf{A}^{-1}\mathbf{B}\,\mathbf{u}_0, \qquad \mathbf{y}_0 = \mathbf{C}\,\mathbf{x}_0$$

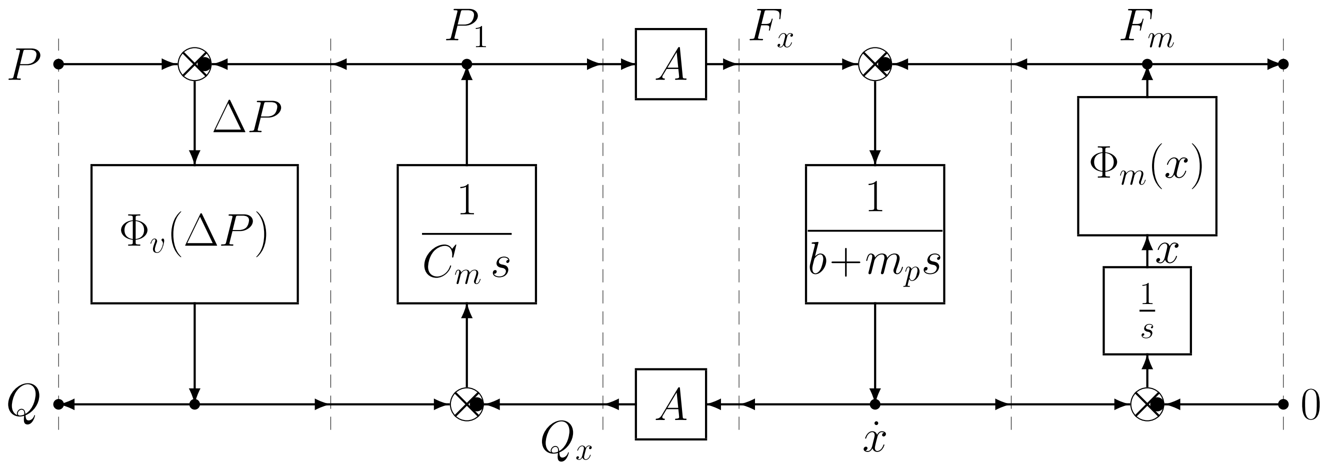
For the considered system it is:

$$\mathbf{x}_0 = \begin{bmatrix} P_1 \\ \dot{x} \\ x \end{bmatrix}_0 = \begin{bmatrix} P \\ 0 \\ \frac{A\,P}{K_m} \end{bmatrix}, \qquad \mathbf{y}_0 = [F_m]_0 = A\,P$$

The transfer function $G(s)$ of a linear system described in the state space can be obtained using the following formula:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

The obtained $G(s)$ is equal to the $G(s)$ obtained using the Mason formula.
If the system is nonlinear, the stability analysis is different. Let us suppose, for example, that, for the considered system, the valve and the spring are described by nonlinear static functions: $\Phi_v(\Delta P)$ e $\Phi_m(x)$.




The state space description of the nonlinear system is now the following:

$$\begin{bmatrix} \dot{P}_1 \\ \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -\dfrac{\Phi_v(P_1 - P)}{C_m} - \dfrac{A\,\dot{x}}{C_m} \\ \dfrac{A\,P_1}{m_p} - \dfrac{b\,\dot{x}}{m_p} - \dfrac{\Phi_m(x)}{m_p} \\ \dot{x} \end{bmatrix}, \qquad F_m = \Phi_m(x)$$

Setting $x_1 = P_1$, $x_2 = \dot{x}$, $x_3 = x$, $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^T$, $\mathbf{u} = u = P$ and $\mathbf{y} = F_m$, the state space equations can be rewritten as follows:

$$\left\{\begin{array}{l} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \frac{\Phi_v(u-x_1)}{C_m} - \frac{A\,x_2}{C_m} \\ \frac{A\,x_1}{m_p} - \frac{b\,x_2}{m_p} - \frac{\Phi_m(x_3)}{m_p} \\ x_2 \end{bmatrix}}_{\mathbf{f}(\mathbf{x},\mathbf{u})} \\ \mathbf{y} = \underbrace{\begin{bmatrix} \Phi_m(x_3) \end{bmatrix}}_{\mathbf{h}(\mathbf{x})} \end{array}\right. \qquad \leftrightarrow \qquad \left\{\begin{array}{lcl} \dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x},\,\mathbf{u}) \\ \mathbf{y} & = & \mathbf{h}(\mathbf{x}) \end{array}\right.$$

where $\mathbf{h}(\mathbf{x})$ is a nonlinear function of the state vector $\mathbf{x}$. In the general case the function $\mathbf{h}(\mathbf{x},\,\mathbf{u})$ also depends on the input vector $\mathbf{u}$. Also in this case the final equilibrium point $\bar{\mathbf{x}}$ associated to the constant input $\mathbf{u} = \bar{\mathbf{u}}$ can be determined imposing $\dot{\mathbf{x}} = 0$:

$$\mathbf{f}(\bar{\mathbf{x}},\,\bar{\mathbf{u}}) = 0 \quad \rightarrow \quad \left\{\begin{array}{l} \frac{\Phi_v(\bar{u}-\bar{x}_1)}{C_m} = 0 \\ \frac{A\,\bar{x}_1}{m_p} - \frac{\Phi_m(\bar{x}_3)}{m_p} = 0 \\ \bar{x}_2 = 0 \end{array}\right. \quad \rightarrow \quad \left\{\begin{array}{l} \bar{x}_1 = \bar{u} - \Phi_v^{-1}(0) \\ \bar{x}_2 = 0 \\ \bar{x}_3 = \Phi_m^{-1}(A\,\bar{x}_1) \end{array}\right.$$

For nonlinear systems there may be several equilibrium points (also called "working points") associated with the same input vector $\mathbf{u}$. A nonlinear system can be "linearized" in the neighborhood of an equilibrium point $\bar{\mathbf{x}}$ as follows:

$$\left\{\begin{array}{lcl} \dot{\mathbf{x}} & = & \left.\frac{\partial \mathbf{f}(\mathbf{x},\,\mathbf{u})}{\partial\,\mathbf{x}^T}\right|_{(\bar{\mathbf{x}},\bar{\mathbf{u}})} (\mathbf{x} - \bar{\mathbf{x}}) + \left.\frac{\partial \mathbf{f}(\mathbf{x},\,\mathbf{u})}{\partial\,\mathbf{u}^T}\right|_{(\bar{\mathbf{x}},\bar{\mathbf{u}})} (\mathbf{u} - \bar{\mathbf{u}}) \\ \mathbf{y} & = & \bar{\mathbf{y}} + \left.\frac{\partial \mathbf{h}(\mathbf{x})}{\partial\,\mathbf{x}^T}\right|_{(\bar{\mathbf{x}})} (\mathbf{x} - \bar{\mathbf{x}}) \end{array}\right.$$

Setting $\tilde{\mathbf{x}} = \mathbf{x} - \bar{\mathbf{x}}$, $\tilde{\mathbf{y}} = \mathbf{y} - \bar{\mathbf{y}}$ and $\tilde{\mathbf{u}} = \mathbf{u} - \bar{\mathbf{u}}$ one obtains the following linearized dynamic model:

$$\left\{\begin{array}{lcl} \dot{\tilde{\mathbf{x}}} & = & \mathbf{A}\,\tilde{\mathbf{x}} + \mathbf{B}\,\tilde{\mathbf{u}} \\ \tilde{\mathbf{y}} & = & \mathbf{C}\,\tilde{\mathbf{x}} \end{array}\right.$$

where

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}(\mathbf{x},\,\mathbf{u})}{\partial\,\mathbf{x}^T}\right|_{(\bar{\mathbf{x}},\bar{\mathbf{u}})}, \qquad \mathbf{B} = \left.\frac{\partial \mathbf{f}(\mathbf{x},\,\mathbf{u})}{\partial\,\mathbf{u}^T}\right|_{(\bar{\mathbf{x}},\bar{\mathbf{u}})}, \qquad \mathbf{C} = \left.\frac{\partial \mathbf{h}(\mathbf{x})}{\partial\,\mathbf{x}^T}\right|_{(\bar{\mathbf{x}})}$$

For the considered system it is:

$$\mathbf{A} = \begin{bmatrix} -\frac{K_v}{C_m} & -\frac{A}{C_m} & 0 \\ \frac{A}{m_p} & -\frac{b}{m_p} & -\frac{K_m}{m_p} \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \frac{K_v}{C_m} \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & K_m \end{bmatrix}$$

where

$$K_v = -\frac{\partial \Phi_v(u - x_1)}{\partial\, x_1}\bigg|_{(\bar{x}_1, \bar{u})} = \frac{\partial \Phi_v(u - x_1)}{\partial\, u}\bigg|_{(\bar{x}_1, \bar{u})} = \frac{\partial \Phi_v(w)}{\partial\, w}\bigg|_{w = \Phi_v^{-1}(0)}$$

$$K_m = \frac{\partial \Phi_m(x_3)}{\partial\, x_3}\bigg|_{x_3 = \bar{x}_3} \qquad \text{dove} \qquad \bar{x}_3 = \Phi_m^{-1}(A\,(\bar{u} - \Phi_v^{-1}(0)))$$

This obtained linearized model is equal to the same linear dynamic model initially considered.