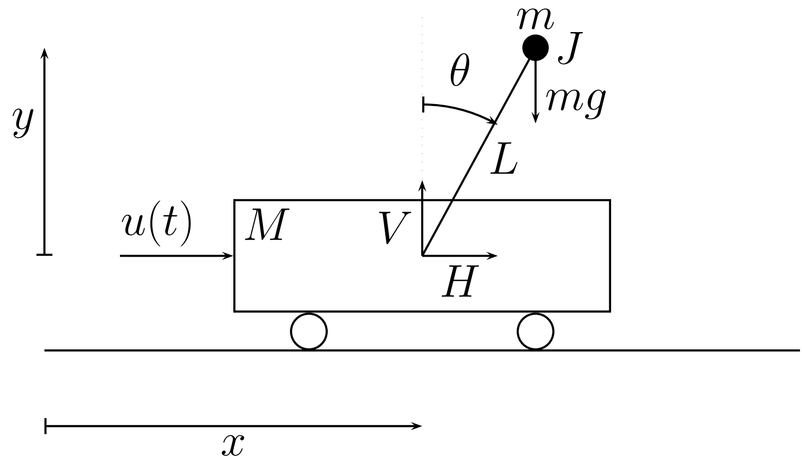
Inverted pendulum control design

Example. Let us consider the following physical system:



Design a dynamic feedback control which stabilizes the system in the vicinity of the point $\theta = 0$.

Let us use the auxiliary variables V and H to describe the interaction between the pole and the cart. The dynamic equations of the system are the following:

$$\text{Pole}: \quad \begin{cases} H(t) = m\frac{d^2}{dt^2}(x + L\sin\theta) \\ V(t) - mg = m\frac{d^2}{dt^2}(L\cos\theta) \\ J\ddot{\theta} = VL\sin\theta - HL\cos\theta \end{cases}$$

$$\text{Cart}: \quad M\ddot{x} = u(t) - H(t)$$

Neglecting the force of the pole acting on the cart, $H(t) \simeq 0$, the acceleration of the cart $\ddot{x}$ is proportional to the input force $u(t)$:

$$H(t) \simeq 0 \qquad \rightarrow \qquad \ddot{x} = \frac{u(t)}{M}$$

Eliminating the auxiliary variables H and V from the system's equations, one obtains a second order differential equation:

$$(J + mL^2)\ddot{\theta} = mgL\sin\theta - m\ddot{x}L\cos\theta$$

which describes the dynamics of the pole around the hinge point.

Making explicit the variable $\ddot{\theta}$ and substituting $u(t)/M$ in place of $\ddot{x}$, one obtains:

$$\begin{aligned}\ddot{\theta} &= \frac{mL}{(J+mL^2)} g\sin\theta - \frac{mL}{(J+mL^2)}\frac{u}{M}\cos\theta \\ &= \alpha g \sin\theta - \frac{\alpha}{M} u\cos\theta\end{aligned}$$

where:

$$\alpha = \frac{mL}{(J+mL^2)}$$

Using the following state vector:

$$\mathbf{x} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x_1 = \theta, \qquad x_2 = \dot{\theta}, \qquad y = \theta = x_1$$

the system can be described in the state space as follows:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) \qquad \leftrightarrow \qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \alpha g \sin x_1 - \frac{\alpha}{M} u \cos x_1 \end{cases}$$

When $u = 0$, the equilibrium points of the system are $(x_1,\, x_2) = (k\pi,\, 0)$ with $k \in \mathcal{Z}$. Linearizing in the vicinity of the equilibrium point $(x_1,\, x_2) = (0,\, 0)$ and $u = 0$, one obtains:

$$\begin{aligned}\dot{\mathbf{x}} &= \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{(0,0)} \mathbf{x} + \left[\frac{\partial \mathbf{f}}{\partial u}\right]_{(0,0)} u \\ &= \begin{bmatrix} 0 & 1 \\ \alpha g \cos x_1 + \frac{\alpha}{M} u \sin x_1 & 0 \end{bmatrix}_{(0,0)} \mathbf{x} + \begin{bmatrix} 0 \\ -\frac{\alpha}{M}\cos x_1 \end{bmatrix}_{(0,0)} u \\ &= \begin{bmatrix} 0 & 1 \\ \alpha g & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -\frac{\alpha}{M} \end{bmatrix} u = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{C}\,\mathbf{x}\end{aligned}$$

The eigenvalues $s_{1,2}$ of matrix $\mathbf{A}$ are real:

$$\Delta_A(s) = s^2 - \alpha g = (s + \sqrt{\alpha g})(s - \sqrt{\alpha g}) \qquad \rightarrow \qquad s_{1,2} = \pm\sqrt{\alpha g}$$

<u>*Static "output" feedback*</u>. Setting:

$$u(t) = \bar{k}\, y = \bar{k}\, \mathbf{C}\, \mathbf{x} = \begin{bmatrix} \bar{k} & 0 \end{bmatrix} \mathbf{x}$$

the matrix of the feedback system is:

$$\mathbf{A} + \mathbf{B}\bar{k}\mathbf{C} = \begin{bmatrix} 0 & 1 \\ \alpha g - \bar{k}\frac{\alpha}{M} & 0 \end{bmatrix}$$

The characteristic polynomial of matrix $\mathbf{A} + \mathbf{B}\bar{k}\mathbf{C}$ is:

$$\Delta_{\mathbf{A}+\mathbf{B}\bar{k}\mathbf{C}}(s) = s^2 - \left(\alpha g - \bar{k}\frac{\alpha}{M}\right)$$

When $\bar{k}$ is large, the eigenvalues are imaginary. When $\bar{k}$ is small, the two eigenvalues are real and one of the two eigenvalues is positive. It follows that the system cannot be stabilized using a static output feedback.

<u>*Static "state" feedback*</u>. The reachability matrix of the system is invertible:

$$\mathcal{R}^+ = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\alpha}{M} \\ -\frac{\alpha}{M} & 0 \end{bmatrix}$$

and therefore using a static state feedback $\mathbf{u} = \mathbf{K}\mathbf{x}$ it is possible to choose arbitrarily the eigenvalues λ_1 and λ_2 of the feedback system. Let $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ be the gain feedback vector. The characteristic polynomials of matrix $\mathbf{A}$ and matrix $\mathbf{A} + \mathbf{BK}$ are:

$$\Delta_{\mathbf{A}}(s) = s^2 - \alpha g, \qquad \Delta_{\mathbf{A}+\mathbf{BK}}(s) = s^2 - (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2$$

The design of the gain vector $\mathbf{K}$ can be done as follows:

$$\begin{aligned}
\mathbf{K} &= \mathbf{K}_c \left(\mathcal{R}^+ (\mathcal{R}_c^+)^{-1}\right)^{-1} = \mathbf{K}_c \left(\begin{bmatrix} 0 & -\frac{\alpha}{M} \\ -\frac{\alpha}{M} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} -\alpha g - \lambda_1\lambda_2 & (\lambda_1 + \lambda_2) \end{bmatrix} \begin{bmatrix} -\frac{M}{\alpha} & 0 \\ 0 & -\frac{M}{\alpha} \end{bmatrix} \\
&= \begin{bmatrix} \frac{M}{\alpha}(\alpha g + \lambda_1\lambda_2) & -\frac{M}{\alpha}(\lambda_1 + \lambda_2) \end{bmatrix}
\end{aligned}$$

<u>*Feedback state observer*</u>. The observability matrix is full rank:

$$\mathcal{O}^- = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and therefore it is possible to choose arbitrarily the eigenvalues β_1 and β_2 of the feedback state observer: $\dot{\hat{\mathbf{x}}} = (\mathbf{A}+\mathbf{LC})\hat{\mathbf{x}}+\mathbf{Bu}-\mathbf{Ly}$. Let $\mathbf{L} = \begin{bmatrix} l_1 & l_2 \end{bmatrix}^T$ be the observer gain vector. The characteristic polynomials of matrix $\mathbf{A}$ and matrix $\mathbf{A}+\mathbf{LC}$ are:

$$\Delta_{\mathbf{A}}(s) = s^2 - \alpha g, \qquad \Delta_{\mathbf{A}+\mathbf{LC}}(s) = s^2 - (\beta_1+\beta_2)s + \beta_1\beta_2$$

The design of the gain vector $\mathbf{L}$ can be done as follows:

$$\begin{aligned} \mathbf{L} &= \left\{(\mathcal{O}_c^-)^{-1}\mathcal{O}^-\right\}^{-1}\mathbf{L}_c = \left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}^{-1}\mathbf{L}_c \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -\alpha g - \beta_1\beta_2 \\ \beta_1+\beta_2 \end{bmatrix} = \begin{bmatrix} \beta_1+\beta_2 \\ -\alpha g - \beta_1\beta_2 \end{bmatrix} \end{aligned}$$

<u>*Reduced order observer*</u>. Using the following state space transformation $\mathbf{x} = \bar{\mathbf{P}}\bar{\mathbf{x}}$, one obtains:

$$\bar{\mathbf{P}} = \left[\begin{array}{cc} 0 & 1 \\ \hline 1 & 0 \end{array}\right] = \bar{\mathbf{P}}^{-1} \quad \rightarrow \quad \begin{cases} \dot{\bar{\mathbf{x}}}(t) = \left[\begin{array}{c|c} 0 & \alpha g \\ 1 & 0 \end{array}\right]\bar{\mathbf{x}}(t) + \left[\begin{array}{c} -\frac{\alpha}{M} \\ \hline 0 \end{array}\right]u(t) \\ y(t) = \left[\begin{array}{c|c} 0 & 1 \end{array}\right]\bar{\mathbf{x}}(t) \end{cases}$$

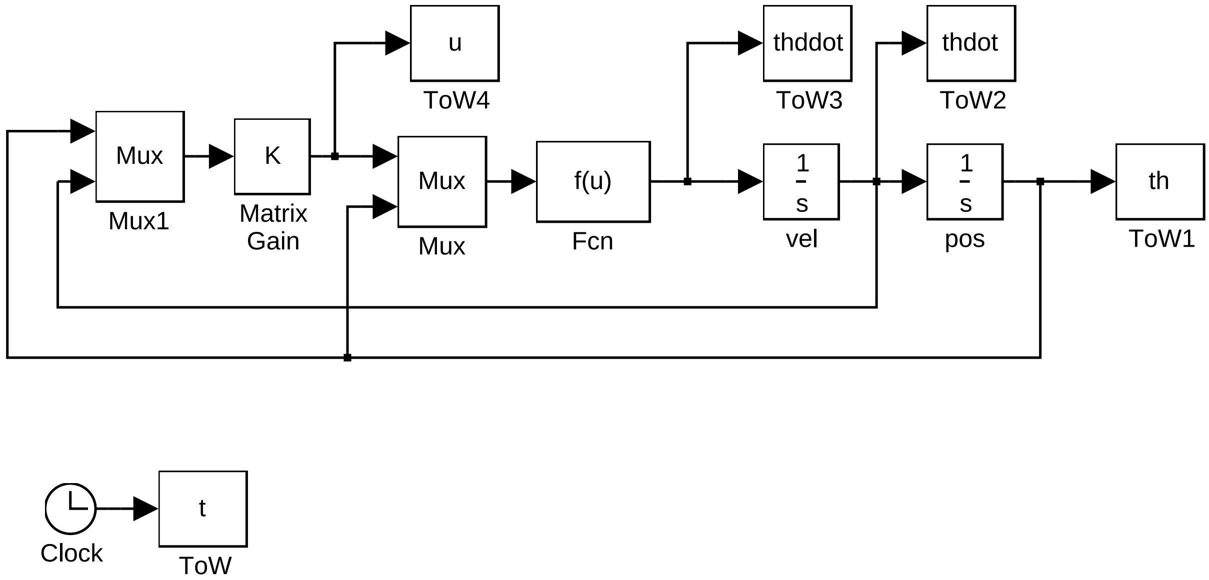
From relation $\mathbf{A}_{11} + \mathbf{LA}_{21} = \mathbf{L} = \beta$ one obtains $\mathbf{L} = \beta$. So, the reduced order observer has the following form:

$$\hat{\mathbf{x}}(t) = \bar{\mathbf{P}}\begin{bmatrix} \hat{v}(t) - \mathbf{L}y(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \hat{v}(t) - \beta y(t) \end{bmatrix}$$

where:

$$\begin{aligned} \dot{\hat{v}}(t) &= (\mathbf{A}_{11}+\mathbf{LA}_{21})\hat{v}(t)+[(\mathbf{A}_{12}+\mathbf{LA}_{22}) - (\mathbf{A}_{11}+\mathbf{LA}_{21})\mathbf{L}]y(t)+(\mathbf{B}_1+\mathbf{LB}_2)u(t) \\ &= \beta\hat{v}(t) + (\alpha g - \beta^2)y(t) - \frac{\alpha}{M}u(t) \end{aligned}$$

Simulation of the feedback system. In Simulink the system dynamics can be described using the following block scheme (pendmdl.mdl):



The system is characterized by the following parameters:

$$M = 2, \qquad m = 0.5, \qquad L = 0.4, \qquad J = 0.02$$

The function "f(u)" is defined as follows:

$$f(u) = \alpha[g \sin(u(2)) - \frac{u(1)}{M} \cos(u(2))]$$

Matrix K is:

$$\left[\ \frac{M}{\alpha}(\alpha g + \lambda_1 \lambda_2) \quad -\frac{M}{\alpha}(\lambda_1 + \lambda_2) \ \right]$$

Matlab file used to simulate the dynamics of the system starting from many different initial conditions (see file "pendulum.m"):

```
 -- Matlab commands ------------------------------------------------------
%%%%%%%%%%%%% Systems's parameters %%%%%%%%%%%%%%%%%
 M=2;               % cart mass
 m=0.5;             % pole mass
 L=0.4;             % pole length
 g=9.81;            % gravity acceleration
 J=0.02;            % pole inertia
 %%%%%%%%%%%%% Simulation and plots %%%%%%%%%%%
 tfin=6;            % Final time
 th00=1;                    % position initial condition
 thdot0=th00;               % velocity initial condition
 lam1=-2+0*1i;              % desired eigenvalue
 lam2=conj(lam1);           % complex coniugate eigenvalue
 alpha=m*L/(J+m*L^2);       % auxiliary variable
 K=M*[alpha*g+lam1*lam2 -(lam1+lam2)]/alpha;    % gain vector
 p=4;
 figure(1); clf                 % open a new figure
 figure(2); clf
 for th0=(-th00:th00/10:th00);  % for each initial condition
   sim('pendulum_mdl',tfin);    % the block scheme 'pendmdl' is simulated
   if (th0>=-0.2)&&(th0<=0.4)
     figure(1)
     plot(t,th); hold on        % the position is plotted
     plot(t,thdot)              % the velocity is plotted
   end
   figure(2)
   plot(th,thdot); hold on      % state space trajectory
   plot(-th,-thdot)
   freccia(th(p),thdot(p),th(p+1),thdot(p+1),th00/30,thdot0/20)
   freccia(-th(p),-thdot(p),-th(p+1),-thdot(p+1),th00/30,thdot0/20)
   %%% freccia() is a function which plots an arrow
 end
 figure(1)
 grid on
 axis([0 tfin -0.5*max(th00,thdot0) max(th00,thdot0)])
 xlabel('time')
 ylabel('th thdot')
 title('traj')
 figure(2)
 if imag(lam1)==0
   plot([-1 1], lam1*[-1 1],':')
 end
 grid on
 axis([-th00 th00 -thdot0 thdot0])
 xlabel('th')
 ylabel('thdot')
 title('phase')
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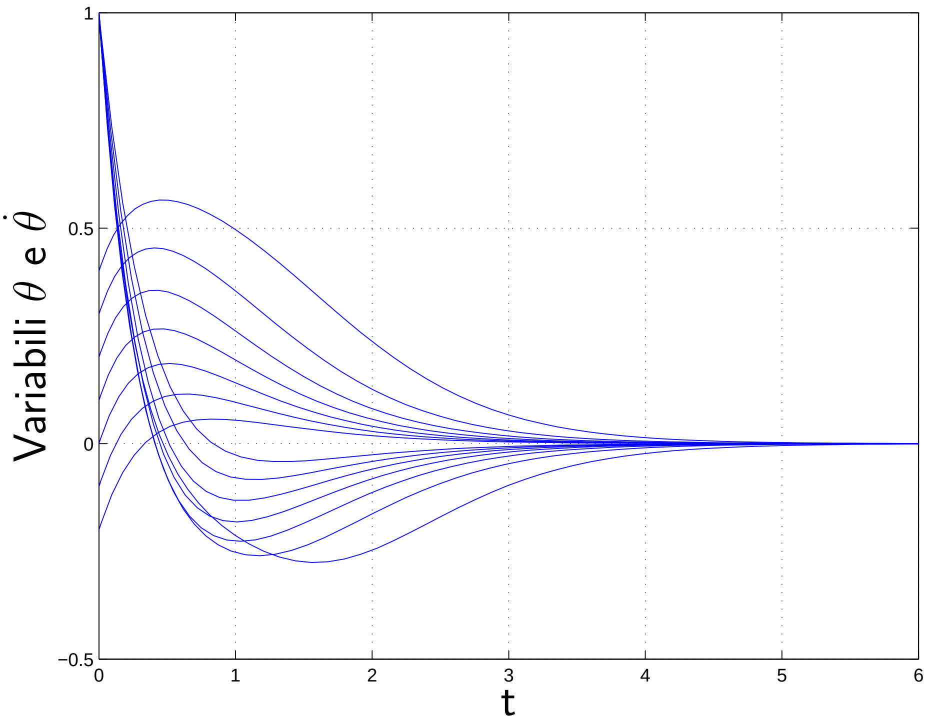
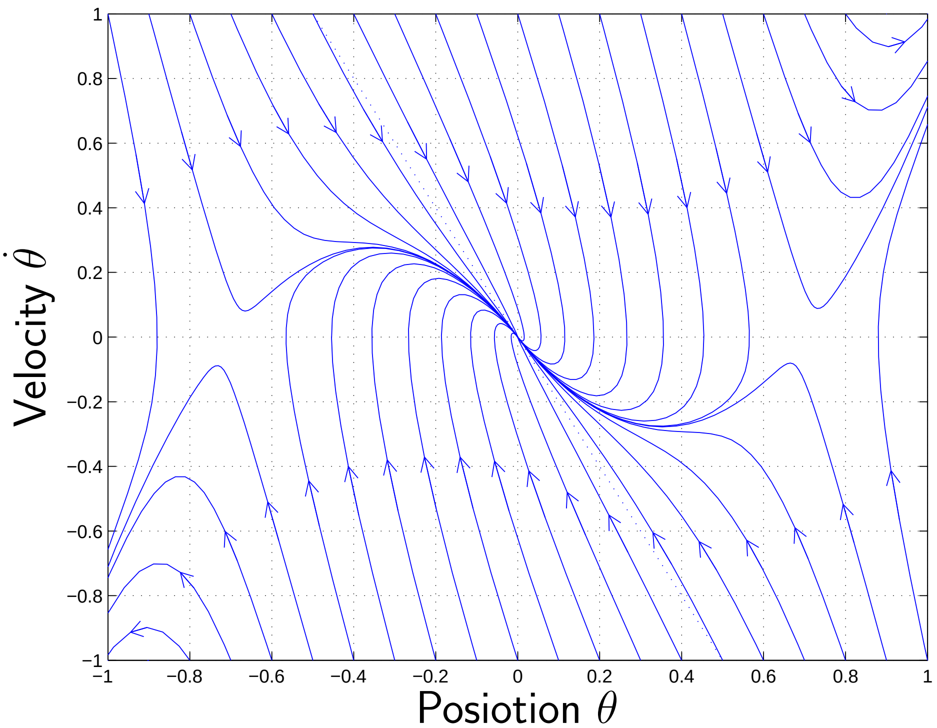
Static state feedback: coincident real eigenvalues

When the eigenvalues are real and coincident, $\lambda_{1,2} = -2$, the time behaviors of variables $\theta(t)$ and $\dot{\theta}(t)$ are aperiodic:

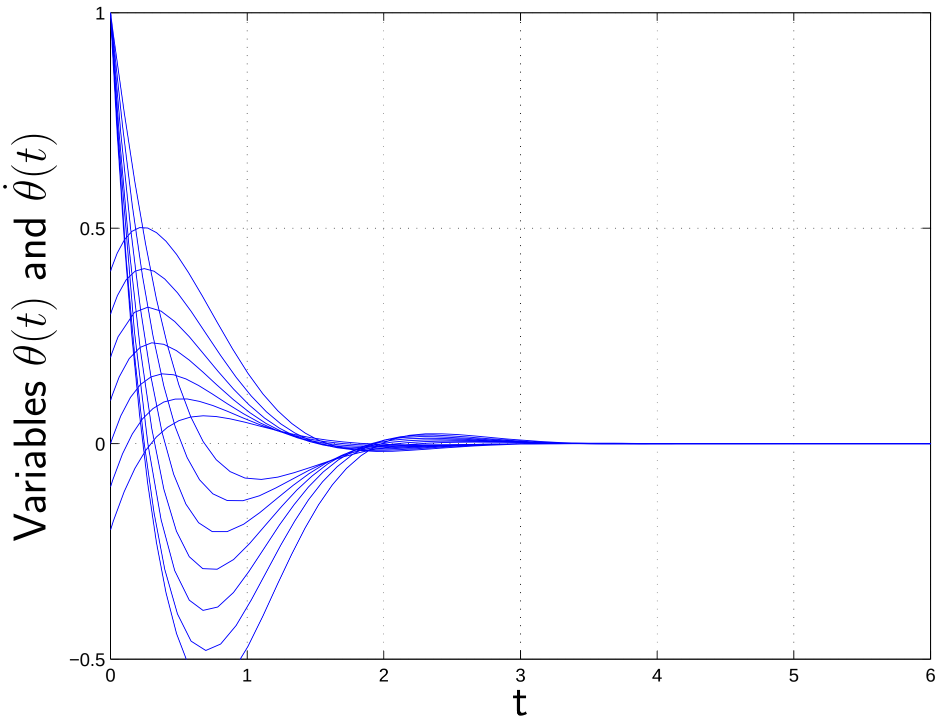


The state space trajectories (stable degenerate node) tends to zero approaching the system eigenvector (dashed line):

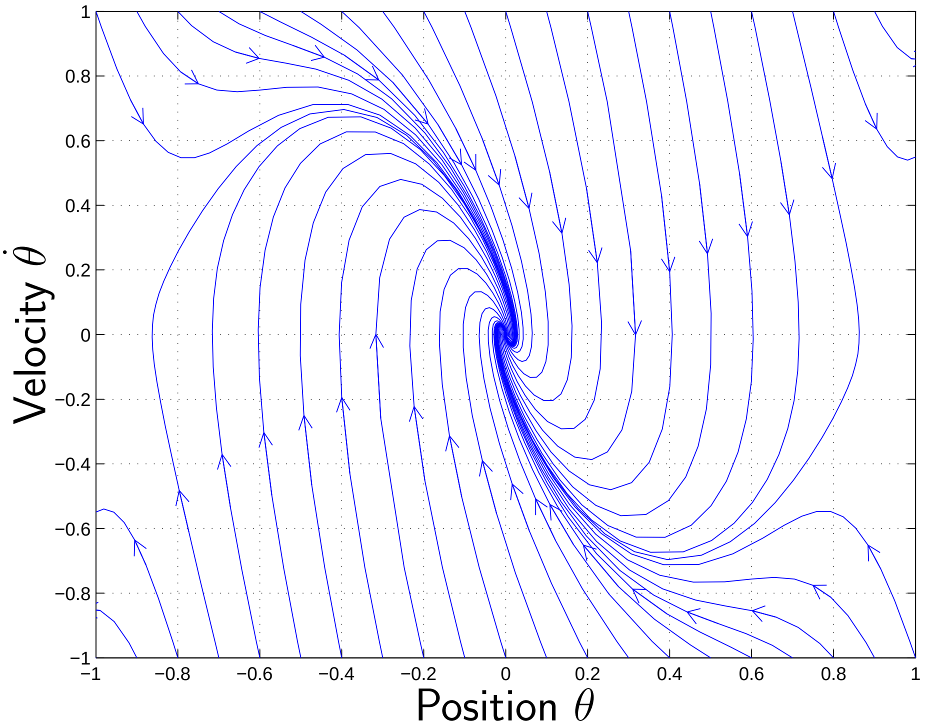


Static state feedback: complex coniugate eigenvalues

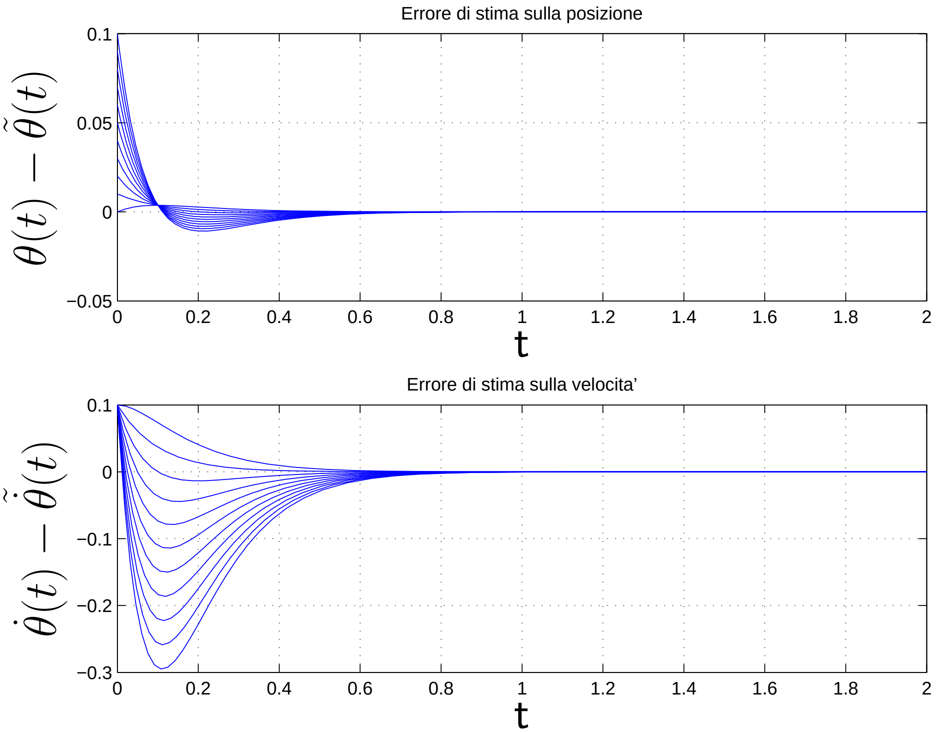
When the eigenvalues are complex coniugate, $\lambda_{1,2} = -2 \pm 2j$, the time behaviors of variables $\theta(t)$ and $\dot{\theta}(t)$ are oscillatory and dumped:



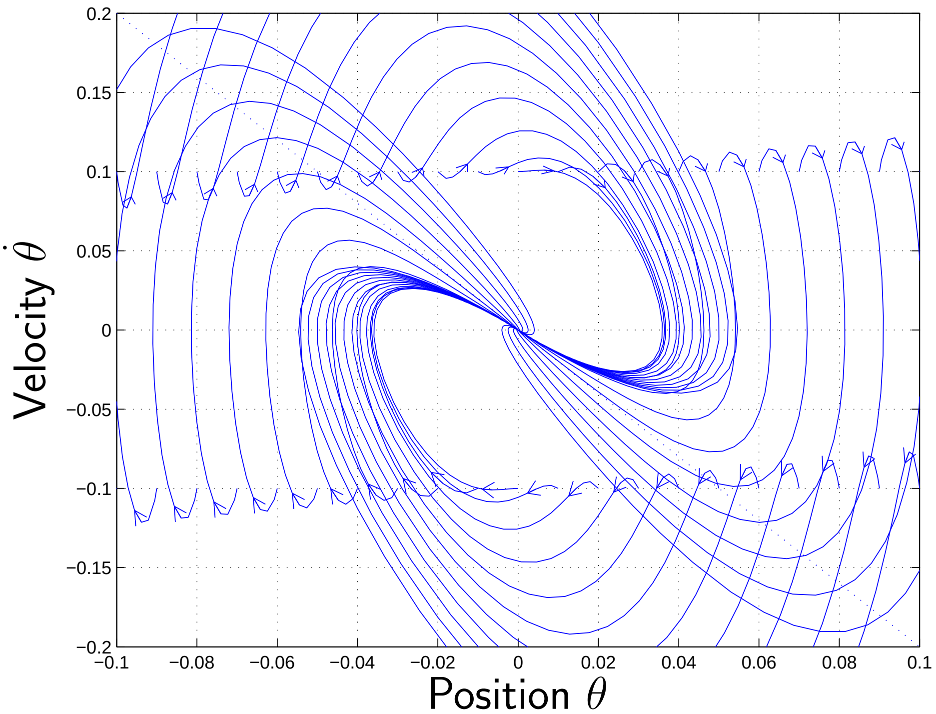
The state space trajectories (stable focus) tend to the origin with a "spiral" type trajectory:



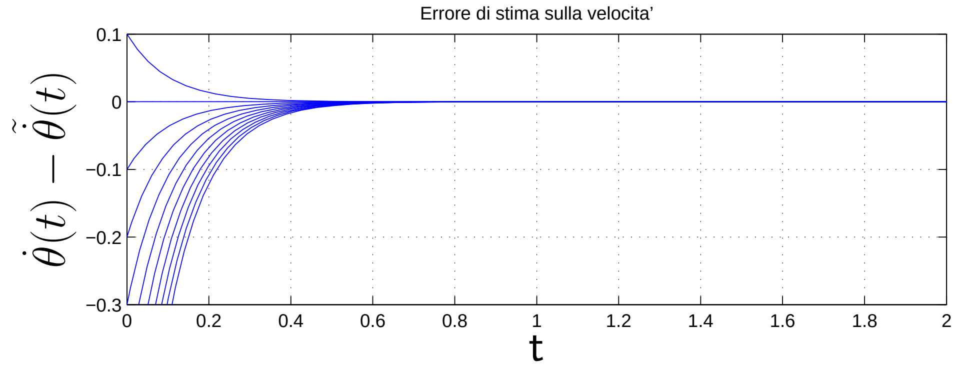
Using the <u>*closed loop estimator*</u>, $\lambda_1 = \lambda_2 = -2$, $\beta_1 = \beta_2 = -10$, one obtains the following time behavior for the velocity estimation error:



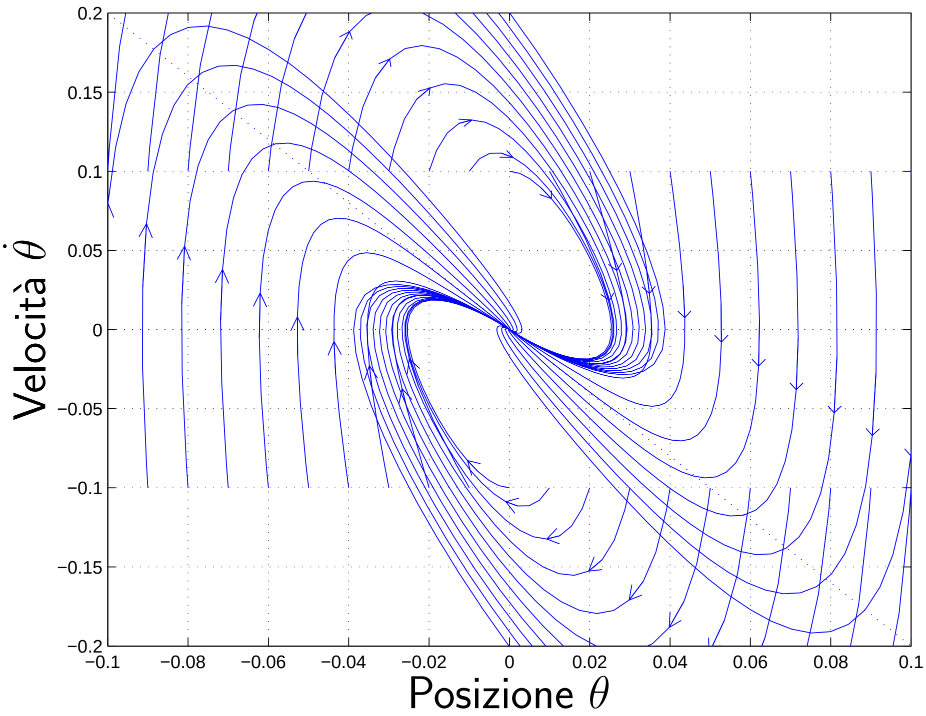
and the following state space trajectories:



Using the *reduced order observer*, $\lambda_1 = \lambda_2 = -2$, $\beta = -10$, one obtains the following time behaviors for the estimation errors:



and the following state space trajectories:



Compared to the previous case, in this case the velocity estimation error tends to zero faster.

For nonlinear system the use of a state observer typically reduces the amplitude of the asymptotic stable region.