Controllability canonical form

- Let us refer to the following linear system $\mathcal{S} = (\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d})$ which is supposed to have only one input ($m = 1$):

$$(1) \quad \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t) + \mathbf{d}\,u(t) \end{cases}$$

- Property. *The system $\mathcal{S} = (\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d})$ is reachable* if and only if *it is algebraically equivalent to a system $\mathcal{S}_c = (\mathbf{A}_c, \mathbf{b}_c, \mathbf{C}_c, \mathbf{d}_c)$ in the following* "controllability canonical form" *characterized by matrices $\mathbf{A}_c$ and $\mathbf{b}_c$ having the following structure:*

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0\ldots & 0 \\ 0 & 0 & 1\ldots & 0 \\ \vdots & \vdots & & \vdots \\ -\alpha_0 & -\alpha_1 & \ldots & -\alpha_{n-1} \end{bmatrix} \qquad \mathbf{b}_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

where the parameters $\alpha_0, \ldots, \alpha_{n-1}$ are the coefficients of the *monic* characteristic polynomial of matrix $\mathbf{A}$:

$$\Delta_{\mathbf{A}}(\lambda) = \lambda^n + \lambda^{n-1}\alpha_{n-1} + \ldots + \alpha_0.$$

- The state space transformation $\mathbf{x} = \mathbf{T}\,\mathbf{x}_c$ which brings system (1) in the controllability canonical form is characterized by the following matrix:

$$\mathbf{T} = \mathcal{R}^+(\mathcal{R}_c^+)^{-1}$$

where $\mathcal{R}^+$ is the reachability matrix of system (1) and where $(\mathcal{R}_c^+)^{-1}$ is a matrix having the following structure:

$$(\mathcal{R}_c^+)^{-1} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \ldots & \ldots & 1 & 0 \\ \alpha_3 & \ldots & \ldots & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \ldots & \ldots & 0 & 0 \\ 1 & 0 & \ldots & \ldots & 0 & 0 \end{bmatrix}$$