Canonical forms

- Let us consider the following SISO dynamic system characterized by matrices $\mathbf{A} \in \mathbf{R}^{n\times n}$, $\mathbf{b} \in \mathbf{R}^{n\times 1}$, $\mathbf{c} \in \mathbf{R}^{1\times n}$ and $d_0 \in \mathbf{R}$:

$$(1) \quad \left\{ \begin{array}{rcl} \dot{\mathbf{x}}(t) & = & \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t) \\ y(t) & = & \mathbf{c}\,\mathbf{x}(t) + d_0\,u(t) \end{array} \right.$$

  The system is continuous-time, linear and time-invariant. The number of parameters of the dynamic part ($\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$) of this system is $n^2+2n+1 = (n+1)^2$: the n^2 parameters of matrix $\mathbf{A}$, the $2n$ parameters of the two vectors $\mathbf{b}$ and $\mathbf{c}$ and coefficient d_0.

- Using proper state space transformations $\mathbf{x} = \mathbf{T}\,\overline{\mathbf{x}}$ it is possible to obtain "different but equivalent" mathematical representations of the given dynamic system. The mathematical representations characterized by the smallest number of nonzero parameters are called **canonical forms**.

- The most interesting canonical forms are the following:
  - Controllability canonical form
  - Observability canonical form
  - Jordan canonical form

- All the canonical forms are characterized by the same number of nonzero parameters: $2\,n+1$.

Controllability canonical form

- Let us consider the following SISO dynamic system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t) \\ y(t) = \mathbf{c}\,\mathbf{x}(t) + d_0\,u(t) \end{cases}$$

- If the reachability matrix $\mathcal{R}^+$ is invertible:

$$\mathcal{R}^+ = \left[\ \mathbf{b},\ \mathbf{A}\mathbf{b},\ \mathbf{A}^2\mathbf{b},\ \ldots,\ \mathbf{A}^{n-1}\mathbf{b}\ \right]$$

the state space transformation $\mathbf{x} = \mathbf{T}_c\,\mathbf{x}_c$ where $\mathbf{T}_c = \mathcal{R}^+\mathbf{T}_\alpha$:

$$\mathbf{T}_c = \underbrace{\left[\ \mathbf{b},\ \mathbf{A}\mathbf{b},\ \mathbf{A}^2\mathbf{b},\ \ldots,\ \mathbf{A}^{n-1}\mathbf{b}\ \right]}_{\mathcal{R}^+}\underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \vdots & \ldots & 1 & 0 \\ \alpha_3 & \vdots & \alpha_{n-1} & \ldots & 0 & 0 \\ \vdots & \alpha_{n-1} & 1 & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}}_{\mathbf{T}_\alpha}$$

brings the system in the following *Controllability canonical form*:

$$\begin{cases} \dot{\mathbf{x}}_c(t) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \ldots & -\alpha_{n-1} \end{bmatrix} \mathbf{x}_c(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \left[\ c_0 \quad c_1 \quad c_2 \quad \ldots \quad c_{n-1}\ \right] \mathbf{x}_c(t) + d_0\,u(t) \end{cases}$$

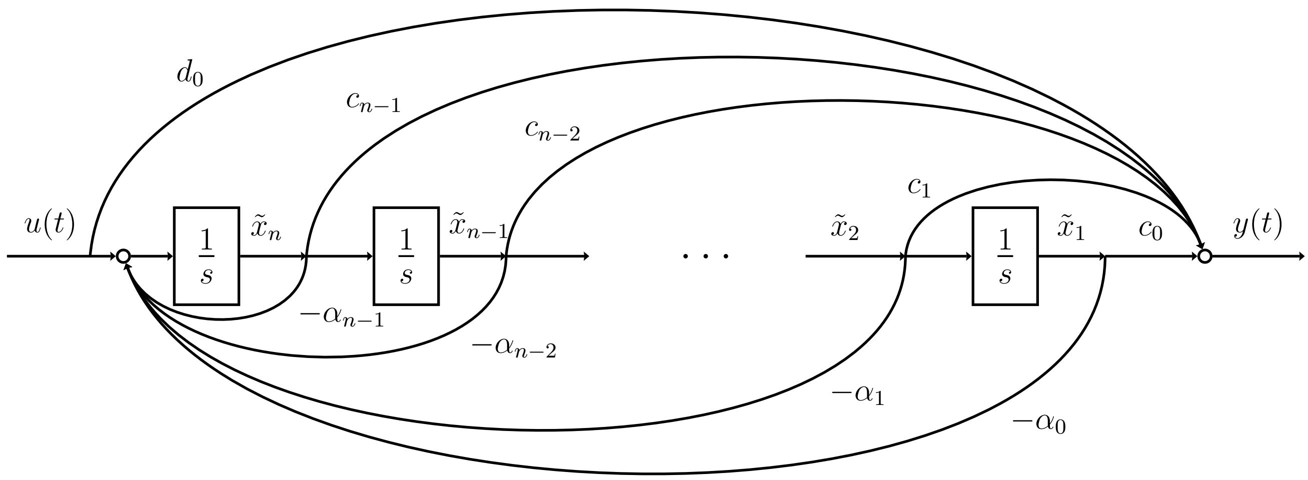
where α_i are the coefficients of the characteristic polynomial of matrix $\mathbf{A}$

$$\Delta_{\mathbf{A}}(s) = \det(s\mathbf{I} - \mathbf{A}) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0$$

The state vector is $\mathbf{x}_c = \left[\ \tilde{x}_1\ \ \tilde{x}_2\ \ \tilde{x}_3\ \ \ldots\ \ \tilde{x}_n\ \right]^T = \left[\ \tilde{x}_1\ \ \dot{\tilde{x}}_1\ \ \ddot{\tilde{x}}_1\ \ \dddot{\tilde{x}}_1\ \ \ldots\ \right]^T$

.

- Block scheme corresponding to the controllability canonical form:



- Using the Mason formula, the determinant $\Delta(s)$ of this block scheme is:

$$\begin{aligned}\Delta(s) &= 1 + \frac{\alpha_{n-1}}{s} + \frac{\alpha_{n-2}}{s^2} + \ldots + \frac{\alpha_0}{s^n}\\ &= \frac{s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0}{s^n}\end{aligned}$$

- The numerator of the Mason formula has the following expression:

$$N(s) = d_0\Delta(s) + \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + c_{n-3}s^{n-3} + \ldots + c_1 s + c_0}{s^n}$$

- The transfer function $G(s) = Y(s)/U(s)$ of the block scheme is the following:

$$G(s) = \frac{N(s)}{\Delta(s)} = d_0 + \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \ldots + c_1 s + c_0}{s^n + \alpha_{n-1}s^{n-1} + \alpha_{n-2}s^{n-2} + \ldots + \alpha_1 s + \alpha_0}$$

- Note that there is a direct biunivocal correspondence between the coefficients c_i and α_i of the transfer function $G(s)$ and the coefficients of the system in the controllability canonical form.

Observability canonical form

- Let us consider the following SISO dynamic system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t) \\ y(t) = \mathbf{c}\,\mathbf{x}(t) + d_0\,u(t) \end{cases}, \qquad \mathcal{O}^- = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \vdots \\ \mathbf{cA}^{n-1} \end{bmatrix}$$

- If the observability matrix $\mathcal{O}^-$ is invertible, then the state space transformation $\mathbf{x} = \mathbf{T}_o\,\mathbf{x}_o$ where:

$$\mathbf{T}_o = \left\{ \underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \dots & \dots & 1 & 0 \\ \alpha_3 & \dots & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1} & 1 & \dots & \dots & 0 & 0 \\ 1 & 0 & \dots & \dots & 0 & 0 \end{bmatrix}}_{\mathbf{T}_\alpha} \underbrace{\begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \mathbf{cA}^2 \\ \dots \\ \mathbf{cA}^{n-1} \end{bmatrix}}_{\mathcal{O}^-} \right\}^{-1}$$

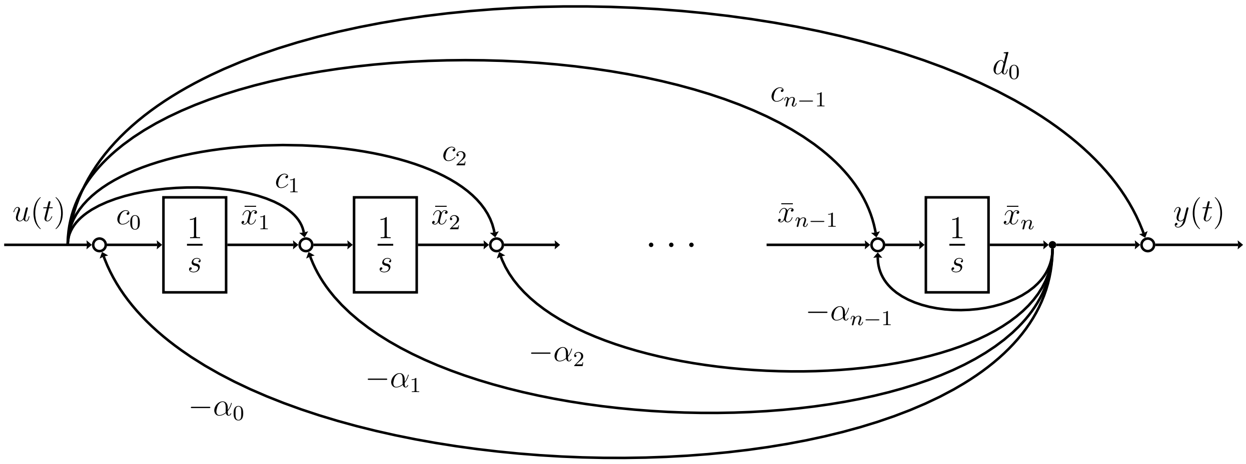
brings the system in the following *observability canonical form*:

$$\begin{cases} \dot{\mathbf{x}}_o(t) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -\alpha_0 \\ 1 & 0 & 0 & \dots & 0 & -\alpha_1 \\ 0 & 1 & 0 & \dots & 0 & -\alpha_2 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -\alpha_{n-2} \\ 0 & 0 & 0 & \dots & 1 & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \vdots \\ \bar{x}_{n-1} \\ \bar{x}_n \end{bmatrix} + \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \end{bmatrix} \mathbf{x}_o(t) + d_0\,u(t) \end{cases}$$

where α_i are the coefficients of the characteristic polynomial of matrix $\mathbf{A}$:

$$\Delta_{\mathbf{A}}(s) = \det(s\mathbf{I} - \mathbf{A}) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0$$

- Block scheme corresponding to the observability canonical form:



- Using the Mason formula, the determinant $\Delta(s)$ of this block scheme is:

$$\begin{aligned}\Delta(s) &= 1 + \frac{\alpha_{n-1}}{s} + \frac{\alpha_{n-2}}{s^2} + \ldots + \frac{\alpha_0}{s^n}\\ &= \frac{s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0}{s^n}\end{aligned}$$

- The numerator of the Mason formula has the following expression:

$$N(s) = d_0\Delta(s) + \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + c_{n-3}s^{n-3} + \ldots + c_1 s + c_0}{s^n}$$

- The transfer function $G(s) = Y(s)/U(s)$ is the following:

$$G(s) = \frac{N(s)}{\Delta(s)} = d_0 + \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \ldots + c_1 s + c_0}{s^n + \alpha_{n-1}s^{n-1} + \alpha_{n-2}s^{n-2} + \ldots + \alpha_1 s + \alpha_0}$$

- Note that there is a direct biunivocal correspondence between the coefficients c_i and α_i of the transfer function $G(s)$ and the coefficients of the system in the observability canonical form.

• Computation of the canonical forms in Matlab (Canonical_Forms.m):

```
clc; echo on                    % Delete the page and  activate the commands' echo
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Random definition of a dynamic system of order n
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n=5; Complex_Poles=1;           % System dimension and type of the poles
sigma=-1-rand(1,n)*5;           % Real part of the poles: sigma = [-6 -1]
Apoli=diag(sigma);              % Matrix Apoli
if Complex_Poles;
    for ii=1:2:n-1
        Apoli(ii+1,ii  )=-Apoli(ii+1,ii+1);
        Apoli(ii  ,ii+1)= Apoli(ii+1,ii+1);
        Apoli(ii+1,ii+1)= Apoli(ii ,ii );
    end
end
T=rand(n); A=T\Apoli*T          % Matrix A=inv(T)*Apoli*T of a stable system
B=rand(n,1)                     % Matrix B of a generic system with m=1
C=rand(1,n)                     % Matrix C of a generic system with p=1
Sys=ss(A,B,C,0);                % Sys = continuous-time dynamic system
poli=roots(poly(A))             % Eigenvalues of matrix A
pause; clc                      % Press any key
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Controllability canonical form
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Rpiu=ctrb(A,B);                 % Controllability matrix
T_alpha=Compute_T_alpha(A);     % Matrix T_alpha
Tc=Rpiu*T_alpha;                % Matrix Tc of the controllability canonical form
Ac=Tc\A*Tc                      % System matrix
Bc=Tc\B                         % Input matrix
Cc=C*Tc                         % Output matrix
SysC=ss(Ac,Bc,Cc,0);            % SysC = system in controllability canonical form
roots(poly(Ac))                 % eigenvalues of matrix Ac
pause; clc                      % Press any key
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Observability canonical form
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Omeno=obsv(A,C);                % Observability matrix
To=inv(T_alpha*Omeno);          % Matrix To of the observability canonical form
Ao=To\A*To                      % System matrix
Bo=To\B                         % Input matrix
Co=C*To                         % Output matrix
SysO=ss(Ao,Bo,Co,0);            % SysO = system in observability canonical form
roots(poly(Ao))                 % Eigenvalues of matrix Ao
pause; clc                      % Press any key
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% Step response is the same for the three systems Sys, SysC and SysO
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure(1); clf                  % Open a new figure
[y,t]=step(Sys);                % Step response of the given dynamic system
plot(t,y,'b','LineWidth',1.5)   % Plots the step response in blue line
hold on; grid on                % Hold on the plots and add the grid on the plot
[y,t]=step(SysC);               % Step response of the system in controllability canonical form
plot(t,y,'g--','LineWidth',2)   % Plot the step response a dashed green line
[y,t]=step(SysO);               % Step response of the system in observability canonical form
plot(t,y,'r:','LineWidth',2.5)  % Plot the step response using a red dotted line
H0=-C*inv(A)*B;                 % Guadagno statico G(0) del sistema SISO
plot(t,0*t+H0,'k--')            % Plot the final values of the output y(t)
xlim([0 t(end)])                % Define the limits of the x axis
title('Step responses of the systems Sys, SysC and SysO')    % Figure title
xlabel('Time t [sec]')          % Label of the x axis
ylabel('y(t)'); echo off        % Label of the y axis y(t) e deactivation of the commands' echo
```

- Function `Sys`, `Compute_T_alpha`:

```
-----------------------------------------------------------------------------------------------
function T_alpha = Compute_T_alpha(A)
n=size(A,1);                        % Dimension of matrix A
T_alpha=zeros(n,n);                 % Initialization of matrix T_alpha
alpha=fliplr(poly(A));              % Coefficients alpha of the characteristic polynomial
for ii=1:n
    T_alpha(ii,n+1-ii)=1;           % Unitary value on the transverse diagonal
    for jj=1:n-ii
        T_alpha(ii,jj)=alpha(jj+ii);    % Coefficienti alpha_i
    end
end
return
-----------------------------------------------------------------------------------------------
```

- System `Sys`:

```
--- Matlab output -----------------------------------------------------------------------------
A =                                                         B =
poli =
   -1.5178   -4.3240  -21.2336   10.3578   -6.0690            0.1992          -1.2607 + 5.7861i
   13.2074   11.5035   26.7385   -0.3523   16.0921            0.5896          -1.2607 - 5.7861i
    2.6183    3.8999    0.9749   -1.7145    1.5169            0.5491          -3.1040 + 2.0504i
   -3.1311    1.1282   16.6624  -11.1747    4.2230            0.6020          -3.1040 - 2.0504i
  -11.6894  -11.9699  -10.0695   -2.0044  -11.1338            0.0835          -2.6183

C =
    0.3842    0.4064    0.9693    0.5298    0.2463
-----------------------------------------------------------------------------------------------
```

- System `Sys0` in controllability canonical form:

```
--- Matlab output -----------------------------------------------------------------------------
Ac =
   1.0e+03 *

   -0.0000    0.0010   -0.0000   -0.0000   -0.0000
         0    0.0000    0.0010    0.0000   -0.0000
    0.0000   -0.0000   -0.0000    0.0010   -0.0000
   -0.0000   -0.0000   -0.0000    0.0000    0.0010
   -1.2707   -1.1467   -0.4216   -0.0874   -0.0113

Bc =                                                            poli =

     0                                                      -1.2607 + 5.7861i
     0                                                      -1.2607 - 5.7861i
     0                                                      -3.1040 + 2.0504i
     0                                                      -3.1040 - 2.0504i
     1                                                      -2.6183

Cc =
   1.0e+03 *

    2.7169    1.3021    0.1901    0.0200    0.0012
-----------------------------------------------------------------------------------------------
```

- System `SysO` in observability canonical form:

```
--- Matlab output -------------------------------------------------------------------
Ao =
   1.0e+03 *

    0.0000   -0.0000    0.0000    0.0000   -1.2707
    0.0010   -0.0000   -0.0000    0.0000   -1.1467
    0.0000    0.0010    0.0000    0.0000   -0.4216
    0.0000   -0.0000    0.0010    0.0000   -0.0874
    0.0000   -0.0000   -0.0000    0.0010   -0.0113

Bo =
   1.0e+03 *                                           poli =

    2.7169                                             -1.2607 + 5.7861i
    1.3021                                             -1.2607 - 5.7861i
    0.1901                                             -3.1040 + 2.0504i
    0.0200                                             -3.1040 - 2.0504i
    0.0012                                             -2.6183

Co =
    0.0000   -0.0000    0.0000    0.0000    1.0000
-------------------------------------------------------------------------------------
```

- Transfer function $G(s)$ of the system:

```
--- Matlab output -------------------------------------------------------------------

          1.188 s^4 + 20.03 s^3 + 190.1 s^2 + 1302 s + 2717
Gs =   -------------------------------------------------------
       s^5 + 11.35 s^4 + 87.42 s^3 + 421.6 s^2 + 1147 s + 1271
-------------------------------------------------------------------------------------
```

- Step response of systems `Sys`, `SysC` and `SysO`: