Ackermann formula: proof

- Given two square matrices $\mathbf{A}$ and $\mathbf{B}$, the following relation holds:

$$(\mathbf{A}+\mathbf{B})^k = \mathbf{A}^k + \sum_{h=0}^{k-1} \mathbf{A}^h \mathbf{B} (\mathbf{A}+\mathbf{B})^{k-1-h} \tag{1}$$

- Let $p(\lambda)$ be the monic desired polynomial:

$$p(\lambda) = \sum_{k=0}^{n} d_k\, s^k = s^n + d_{n-1}\, s^{n-1} + \ldots + d_1\, s + d_0$$

where $d_n = 1$.

- The matrix polynomial function $p(\mathbf{A}+\mathbf{b}\mathbf{k}^T)$ can be written in the following form:

$$p(\mathbf{A}+\mathbf{b}\mathbf{k}^T) = \sum_{k=0}^{n} d_k\, (\mathbf{A}+\mathbf{b}\mathbf{k}^T)^k$$

- Using relation (1) with $\mathbf{B} = \mathbf{b}\mathbf{k}^T$ one obtains:

$$\begin{aligned} p(\mathbf{A}+\mathbf{b}\mathbf{k}^T) &= \sum_{k=0}^{n} d_k \left[\mathbf{A}^k + \sum_{h=0}^{k-1} \mathbf{A}^h \mathbf{b}\mathbf{k}^T (\mathbf{A}+\mathbf{b}\mathbf{k}^T)^{k-1-h} \right] \\ &= \sum_{k=0}^{n} d_k\, \mathbf{A}^k + \sum_{k=0}^{n} d_k \sum_{h=0}^{k-1} \mathbf{A}^h \mathbf{b}\mathbf{k}^T (\mathbf{A}+\mathbf{b}\mathbf{k}^T)^{k-1-h} \\ &= p(\mathbf{A}) + \sum_{k=0}^{n} d_k \sum_{h=0}^{k-1} \mathbf{A}^h \mathbf{b}\mathbf{k}^T (\mathbf{A}+\mathbf{b}\mathbf{k}^T)^{k-1-h} \end{aligned}$$

- The Cayley-Hamilton theorem states that matrix polynomial function $p(\mathbf{A}+\mathbf{b}\mathbf{k}^T)$ is zero if the eigenvalues of matrix $\mathbf{A}+\mathbf{b}\mathbf{k}^T$ are equal to the roots of polynomial $p(\lambda)$. The vector $\mathbf{k}^T$ which nullifies function $p(\mathbf{A}+\mathbf{b}\mathbf{k}^T)$ satisfies the following relation:

$$p(\mathbf{A}) + \sum_{k=0}^{n} d_k \sum_{h=0}^{k-1} \mathbf{A}^h \mathbf{b}\mathbf{k}^T (\mathbf{A}+\mathbf{b}\mathbf{k}^T)^{k-1-h} = 0$$

- Multiplying on the left for the inverse of the reachability matrix one obtains:

$$(\mathcal{R}^+)^{-1}p(\mathbf{A}) + \sum_{k=0}^{n} d_k \sum_{h=0}^{k-1} \underbrace{(\mathcal{R}^+)^{-1}\mathbf{A}^h\mathbf{b}}_{\mathbf{e}_{h+1}} \mathbf{k}^T(\mathbf{A}+\mathbf{b}\mathbf{k}^T)^{k-1-h} = 0$$

- Since $(\mathcal{R}^+)^{-1}\mathbf{A}^h\mathbf{b} = \mathbf{e}_{h+1}$, where $\mathbf{e}_{h+1}$ denotes $(h+1)$-th column the n order identity matrix, the above relation simplifies as follows:

$$(\mathcal{R}^+)^{-1}p(\mathbf{A}) + \sum_{k=0}^{n} d_k \sum_{h=0}^{k-1} \mathbf{e}_{h+1}\mathbf{k}^T(\mathbf{A}+\mathbf{b}\mathbf{k}^T)^{k-1-h} = 0$$

- Multiplying on the left for the vector $\mathbf{e}_n^T$ one obtains:

$$\mathbf{e}_n^T(\mathcal{R}^+)^{-1}p(\mathbf{A}) + \sum_{k=0}^{n} d_k \sum_{h=0}^{k-1} \mathbf{e}_n^T\mathbf{e}_{h+1}\mathbf{k}^T(\mathbf{A}+\mathbf{b}\mathbf{k}^T)^{k-1-h} = 0$$

- One can easily verify that the term $\mathbf{e}_n^T\mathbf{e}_{h+1}$ is zero only when $h = n-1$ and $k = n$. So, the above relation simplifies as follows:

$$\mathbf{e}_n^T(\mathcal{R}^+)^{-1}p(\mathbf{A}) + d_n\,\mathbf{k}^T(\mathbf{A}+\mathbf{b}\mathbf{k}^T)^0 = 0$$

- Being $d_n = 1$ and $(\mathbf{A}+\mathbf{b}\mathbf{k}^T)^0 = \mathbf{I}_n$, one obtains:

$$\mathbf{e}_n^T(\mathcal{R}^+)^{-1}p(\mathbf{A}) + \mathbf{k}^T = 0$$

- Setting $\mathbf{q} = \mathbf{e}_n^T$ and putting in evidence the vector $\mathbf{k}^T$ one obtains the following Ackermann formula:

$$\boxed{\mathbf{k}^T = -\mathbf{q}\,(\mathcal{R}^+)^{-1}p(\mathbf{A})}$$

which can be used only if matrix $\mathcal{R}^+$ is squared and invertible, that is only if the system is completely reachable and has only one input.