Sliding Mode Control

Example. Let us consider the following dynamic system:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_0 x_1 - a_1 x_2 + u \end{cases}$$

Analyze the dynamic behavior of the controlled system when the following nonlinear control law is used:

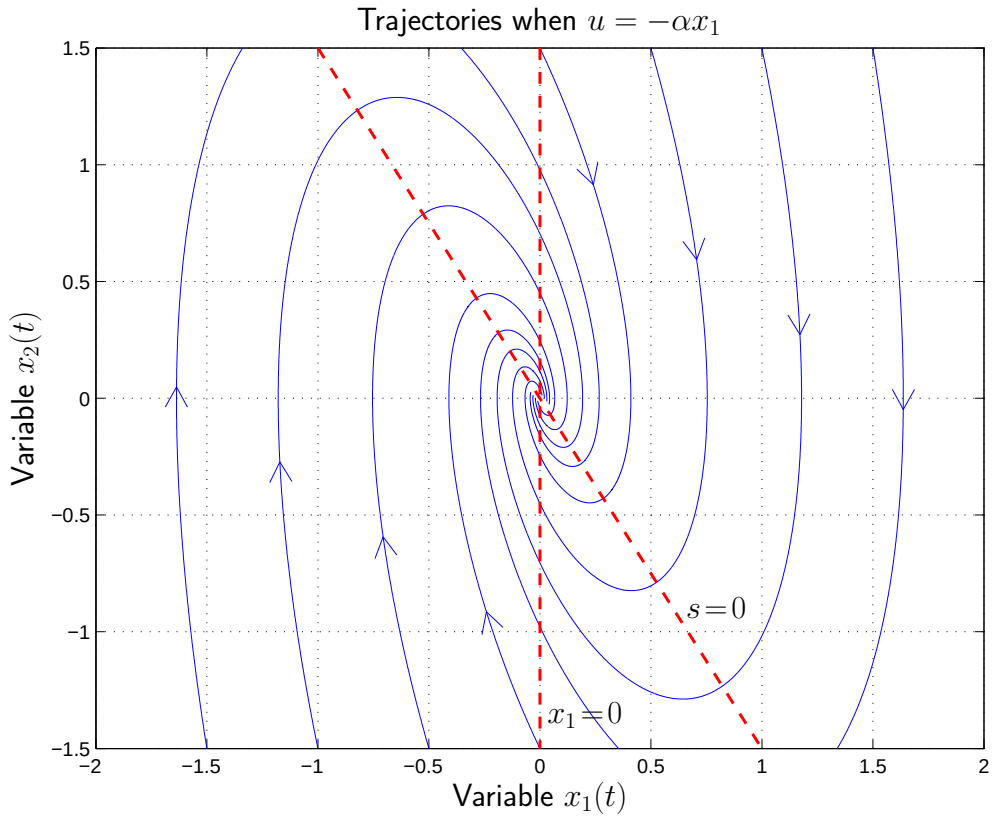
$$u = -\alpha\, x_1\, \text{sign}(x_1 s) \qquad \text{where} \qquad s = c_1 x_1 + x_2$$

Use the following parameters: $a_0 = 1$, $a_1 = 2$, $\alpha = 3$, $c_2 = 1.5$.

Solution. The characteristic polynomial of the non-controlled system is: $p(s) = s^2 + a_1 s + a_0$. The given control law has a "variable structure" because when the function $x_1 s$ changes sign the system changes its dynamic behavior. The characteristic polynomial of the feedback system is:

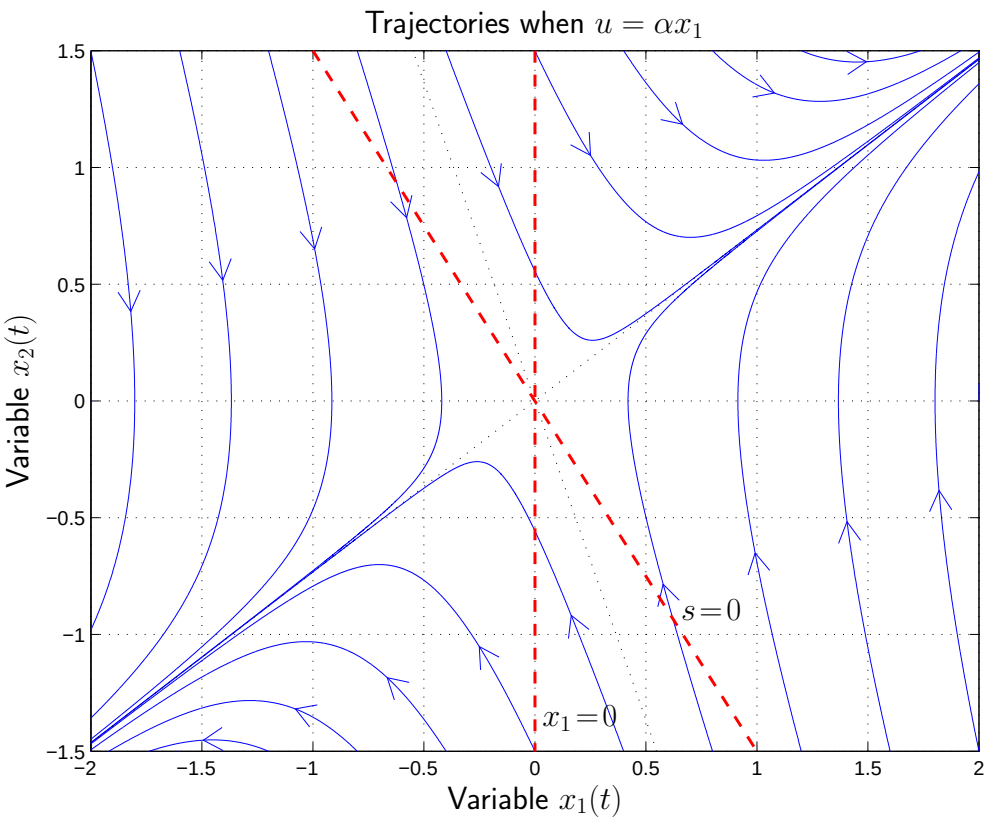
$$p(s) = s^2 + a_1 s + a_0 + \alpha\, \text{sign}(x_1 s).$$

Systems's trajectories when $\text{sign}(x_1 s) > 0$ (stable focus):

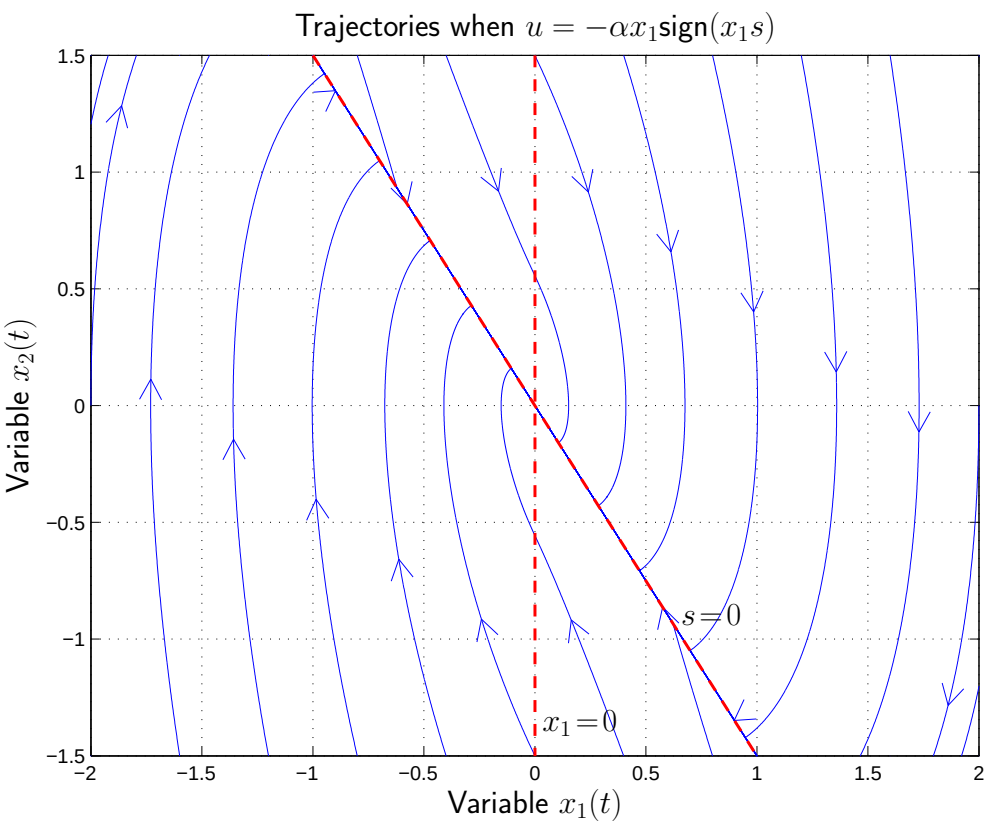


The red dashed lines are the lines along with the function $x_1 s$ changes sign. See files "`Sliding_Example.m`" and "`Sliding_Example_mdl.slx`"

Systems's trajectories when sign$(x_1 s) < 0$ with $\alpha > a_0$ (saddle):

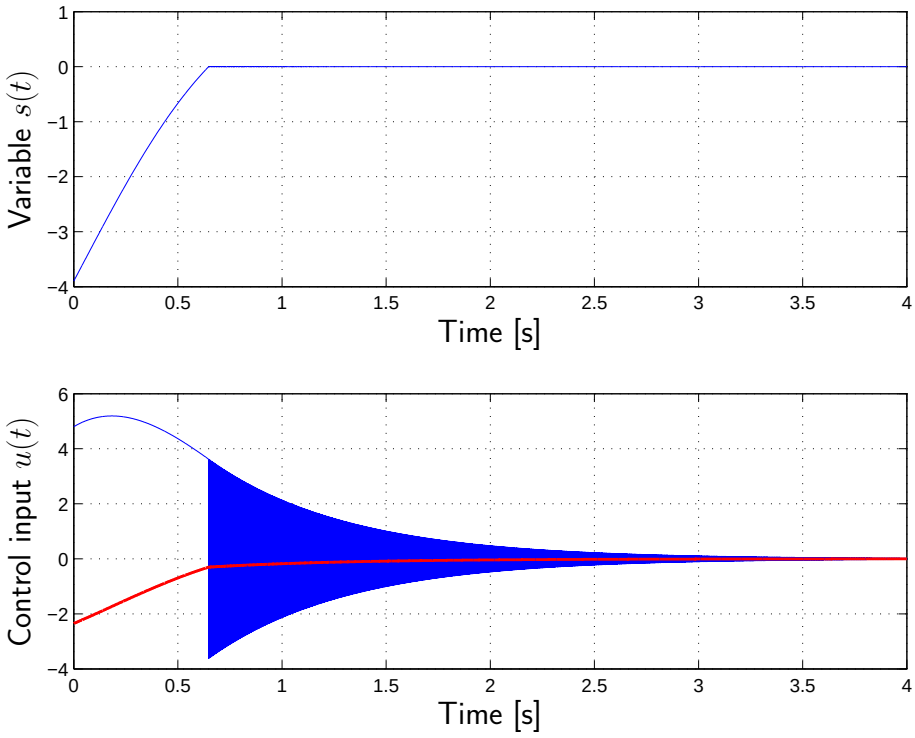


Systems's trajectories when the control law $u = -\alpha x_1$ sign$(x_1 s)$ is used:



The controlled system is asymptotically stable.

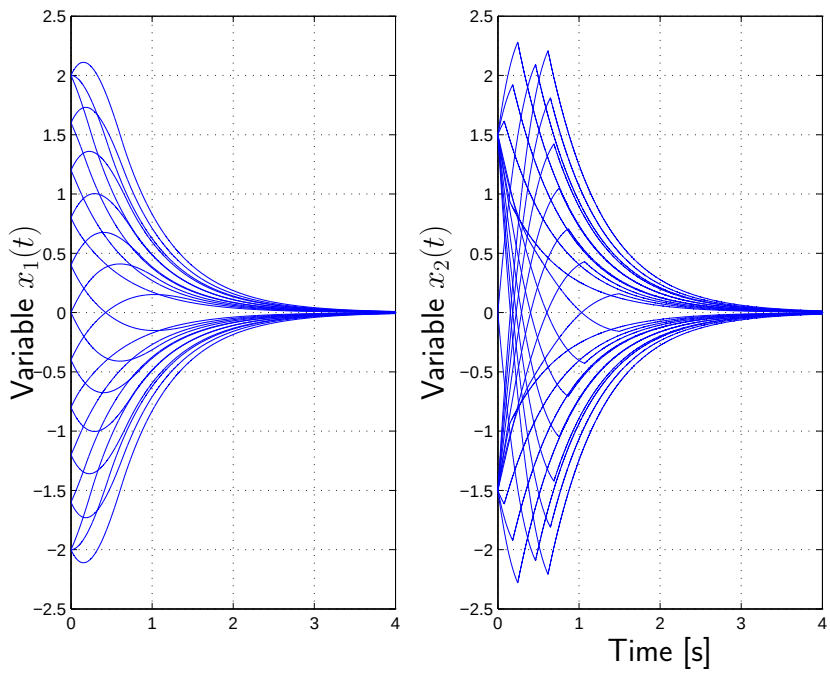
When the trajectories reach the line $s = 0$, that is when $x_2 = -c_1 x_1$, the control signal $u(t)$ starts to switch at "infinite" frequency and the trajectories "slides" along the straight line $s = 0$.



Remembering that $x_2 = \dot{x}_1$, the system dynamics in the "sliding" condition is:

$$s = 0, \qquad \rightarrow \qquad \dot{x}_1 = -c_1 x_1 \qquad \rightarrow \qquad x_1(t) = e^{-c_1 t} x_1(0)$$

The state variables tend to zero with a velocity (i.e. a settling time) which does not depend on the system parameters, but it depends only on the coefficient c_1 of the sliding surface $s = x_2 - c_1 x_1 = 0$.



The control signal $u(t)$ switches at infinite frequency and in each instant the mean value of the control signal $u(t)$ is equal to the "equivalent control u_{eq}" needed to keep the system trajectory on the sliding surface $s = 0$. The equivalent control u_{eq} can be determined imposing $\dot{s} = 0$:

$$\dot{s} = 0 \qquad \rightarrow \qquad c_1\dot{x}_1 + \dot{x}_2 = 0 \qquad \rightarrow \qquad c_1x_2 - a_0x_1 - a_1x_2 + u_{eq} = 0$$

from which one obtains:

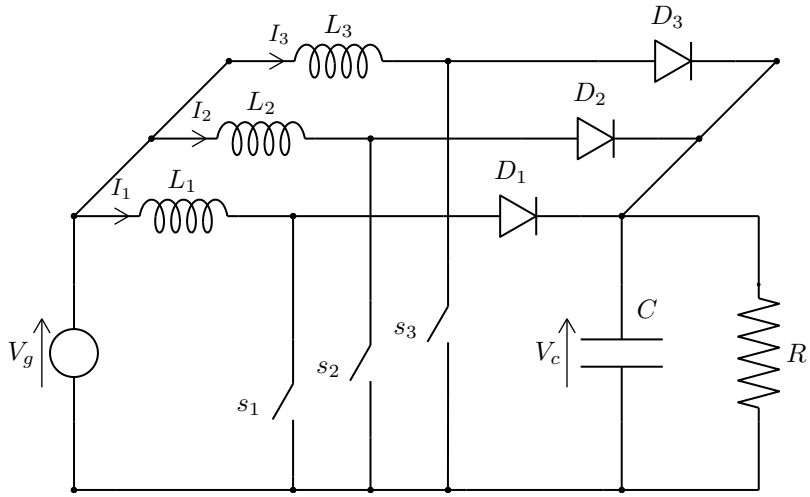
$$u_{eq} = -c_1x_2 + a_0x_1 + a_1x_2$$

The Sliding Mode control technique is characterized by the following features:

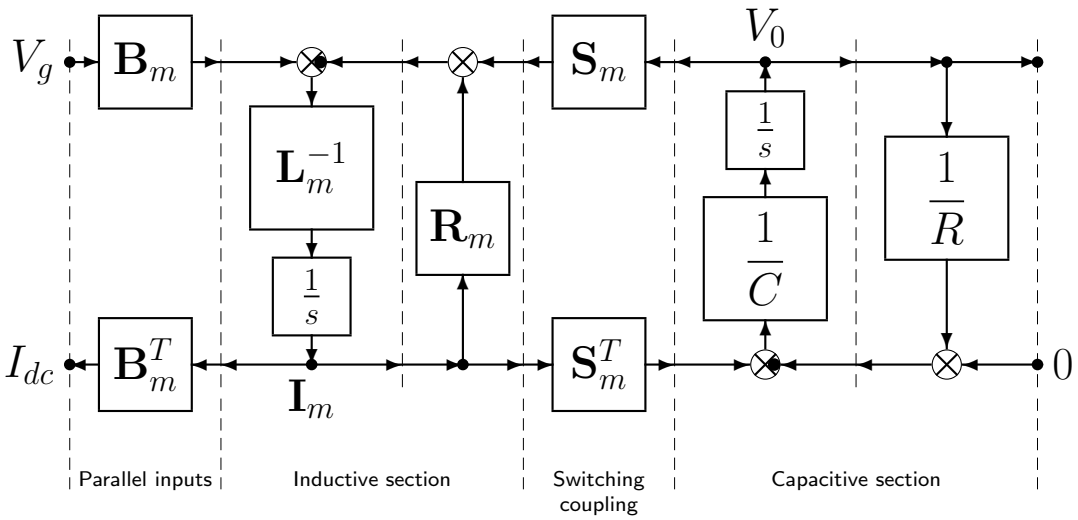
- It can be used only when the control signal $u(t)$ can switch at high frequency. It is suitable, for example, for controlling electric motors where the control signal $u(t)$ is usually a voltage.
- From a practical point of view the switching frequency is always "finite", and therefore the sliding surface $s = 0$ is always tracked with a small error.
- The "high frequency" switching of the control variable $u(t)$ induces an high frequency oscillation within the system called "chattering". It the chattering is not "properly filtered" it can cause vibrations or noise within the system.
- The "external disturbances" which act on the system during the sliding control and which does not saturate the control action *are completely and instantaneously "rejected"*.
- On the sliding surface $s = 0$ the dynamics of the system depends only on the parameters of the sliding surface.

Example: Interleaved Boost Converter

Let us consider the following physical system which describes an "Interleaved Boost Converter with magnetic coupling" ($m = 3$):



The POG scheme of the Boost Converter is the following:



Matrices $\mathbf{B}_m$ $\mathbf{L}_m$, $\mathbf{R}_m$, $\mathbf{S}_m$ and $\mathbf{I}_m$ are defined as follows:

$$\mathbf{B}_m = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad \mathbf{L}_m = \begin{bmatrix} L_1 & M_{12} & M_{13} & \dots & M_{1m} \\ M_{12} & L_2 & M_{23} & \dots & M_{2m} \\ M_{13} & M_{23} & L_3 & \dots & M_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{1m} & M_{2m} & M_{3m} & \dots & L_m \end{bmatrix},$$

and

$$\mathbf{R}_m = \begin{bmatrix} R_1 & 0 & 0 & \dots & 0 \\ 0 & R_2 & 0 & \dots & 0 \\ 0 & 0 & R_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & R_m \end{bmatrix}, \qquad \mathbf{S}_m = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_m \end{bmatrix}, \qquad \mathbf{I}_m = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_m \end{bmatrix}$$

where s_i are the control signals:

$$s_i \in \{0,\ 1\}, \qquad i \in \{1,\ 2,\ 3\}$$

When $s_i = 0$ the switch is "closed" and when $s_i = 1$ the switch is "open". The corresponding state space equations can be written as follows:

$$\underbrace{\begin{bmatrix} \mathbf{L}_m & 0 \\ 0 & C \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{\mathbf{I}}_m \\ \dot{V}_0 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -\mathbf{R}_m & -\mathbf{S}_m \\ \mathbf{S}_m & -\frac{1}{R} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{I}_m \\ V_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{B}_m \\ 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} V_g \end{bmatrix}}_{\mathbf{u}} \tag{1}$$

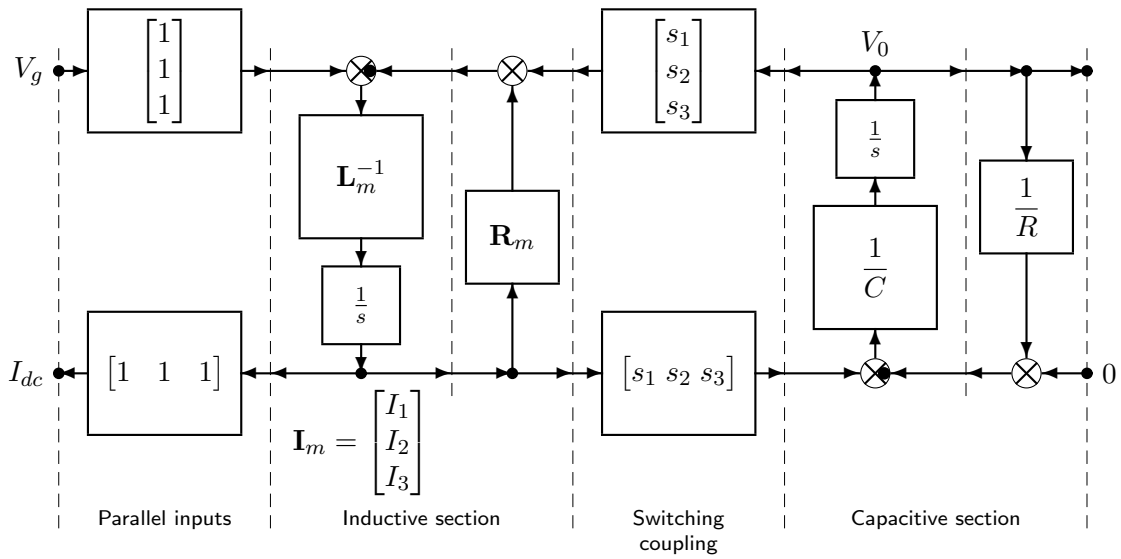
In a compact form it is:

$$\mathbf{L}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$$

The energy E_s stored in the system and the dissipating power P_d can be easily expressed as follows:

$$E_s = \frac{1}{2}\mathbf{x}^T\,\mathbf{L}\,\mathbf{x}, \qquad P_d = \mathbf{x}^T\,\mathbf{A}\,\mathbf{x}$$

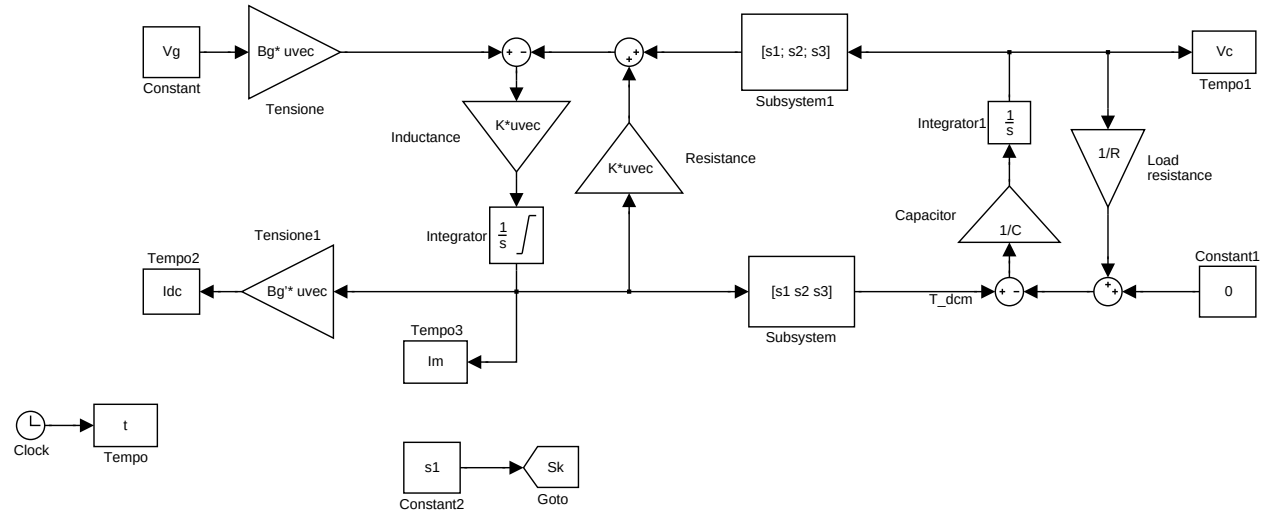
The POG scheme of the Boost Converter when $m = 3$ is the following:



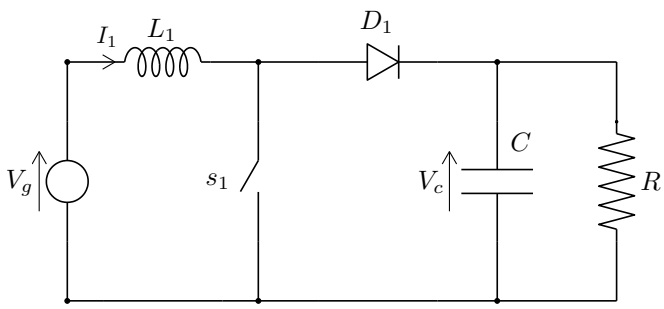
From the POG scheme one obtains the following state space description:

$$\left[\begin{array}{ccc|c} L_1 & M_{12} & M_{13} & 0 \\ M_{12} & L_2 & M_{23} & 0 \\ M_{13} & M_{23} & L_3 & 0 \\ \hline 0 & 0 & 0 & C \end{array}\right] \left[\begin{array}{c} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \\ \hline \dot{V}_0 \end{array}\right] = \left[\begin{array}{ccc|c} -R_1 & 0 & 0 & -s_1 \\ 0 & -R_2 & 0 & -s_2 \\ 0 & 0 & -R_3 & -s_3 \\ \hline s_1 & s_2 & s_3 & -\frac{1}{R} \end{array}\right] \left[\begin{array}{c} I_1 \\ I_2 \\ I_3 \\ \hline V_0 \end{array}\right] + \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ \hline 0 \end{array}\right] \left[V_g\right]$$

The POG scheme can also be "directly" introduced in Simulink:



<u>Case $m = 1$.</u> When $m = 1$ the system has the following structure:



The corresponding state space equations simplify as follows:

$$\begin{cases} L_1 \dot{I}_1 &= -R_1 I_1 - s_1 V_0 + V_g \\ C \dot{V}_0 &= s_1 I_1 - \frac{1}{R} V_0 \end{cases} \tag{2}$$

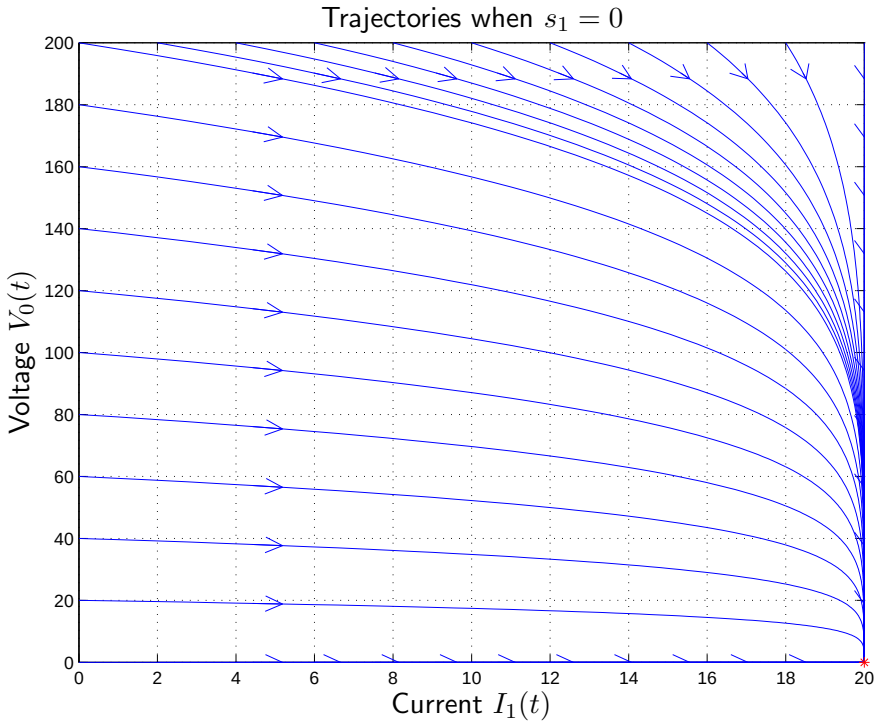
In matrix form the dynamics of the system can be described as follows:

$$\left[\begin{array}{c|c} L_1 & 0 \\ \hline 0 & C \end{array}\right] \left[\begin{array}{c} \dot{I}_1 \\ \hline \dot{V}_0 \end{array}\right] = \left[\begin{array}{c|c} -R_1 & -s_1 \\ \hline s_1 & -\frac{1}{R} \end{array}\right] \left[\begin{array}{c} I_1 \\ \hline V_o \end{array}\right] + \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right] V_g \tag{3}$$

When $s_1 = 0$, the two dynamic elements are "decoupled":

$$\begin{cases} L_1 \dot{I}_1 &= -R_1 I_1 + V_g \\ C \dot{V}_0 &= -\frac{1}{R} V_0 \end{cases} \Rightarrow \begin{cases} I_1(t) &= e^{-\frac{R_1}{L_1} t} I_{10} + \frac{V_g}{R_1} \left[1 - e^{-\frac{R_1}{L_1} t}\right] \\ V_0(t) &= e^{-\frac{t}{RC}} V_0 \end{cases}$$

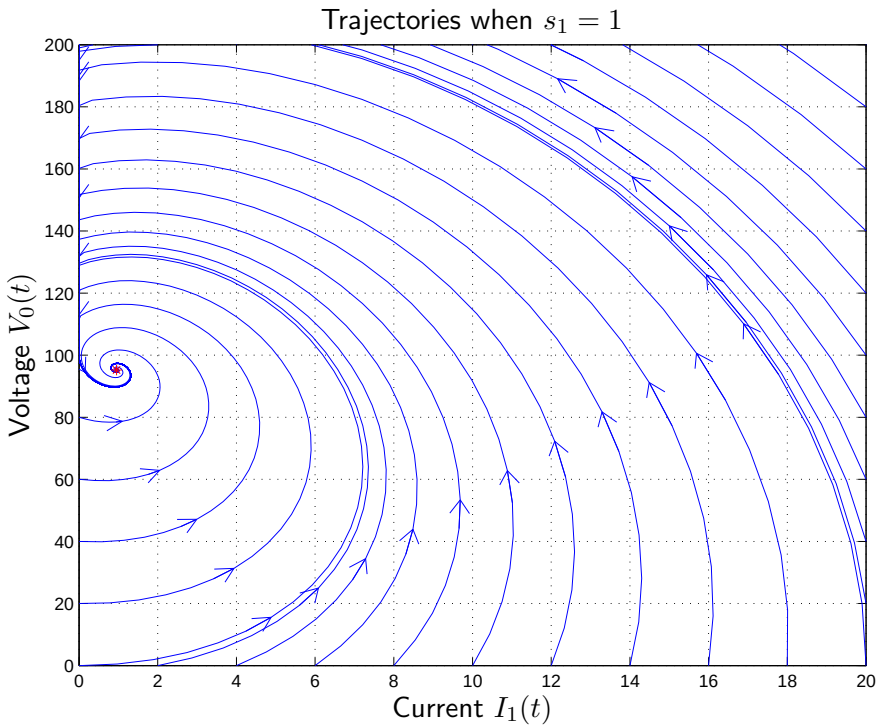
When $s_1 = 0$ the equilibrium point is: $(I_1,\ V_0) = (\frac{V_g}{R_1},\ 0)$. The corresponding state space trajectories are:



When $s_1 = 1$, the two dynamic elements are "coupled":

$$\begin{cases} L_1 \dot{I}_1 &= -R_1 I_1 - V_0 + V_g \\ C\dot{V}_0 &= I_1 - \frac{1}{R} V_0 \end{cases}$$

When $s_1 = 1$ the equilibrium point is $(I_1,\ V_0) = (\frac{V_g}{R_1+R},\ \frac{V_g R}{R_1+R})$ and the state space trajectories (see "`Boost_Example.m`" and "`Boost_Example_mdl.slx`") are:



1) Feedforward control

Let us consider a constant duty cycle: $\bar{s}_1 \in [0,\, 1]$. The corresponding equilibrium point can be determined from (3) imposing $\dot{I}_0$ and $\dot{V}_0 = 0$:

$$\left[\begin{array}{c|c} -R_1 & -\bar{s}_1 \\ \hline \bar{s}_1 & -\frac{1}{R} \end{array}\right] \begin{bmatrix} I_1 \\ V_0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_g = 0 \quad \rightarrow \quad \begin{bmatrix} I_1 \\ V_0 \end{bmatrix} = -\left[\begin{array}{c|c} -R_1 & -\bar{s}_1 \\ \hline \bar{s}_1 & -\frac{1}{R} \end{array}\right]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_g$$

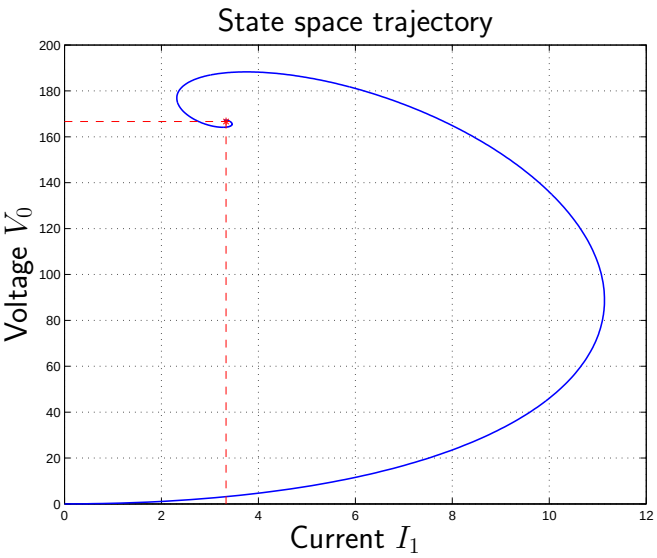
from which one obtains:

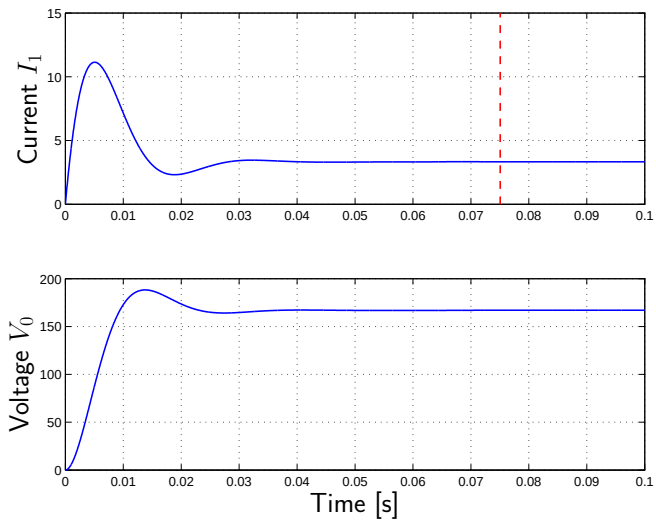
$$\begin{bmatrix} I_1 \\ V_0 \end{bmatrix} = -\frac{R}{R_1 + R\,\bar{s}_1^2} \left[\begin{array}{c|c} -\frac{1}{R} & \bar{s}_1 \\ \hline -\bar{s}_1 & -R_1 \end{array}\right] \begin{bmatrix} V_g \\ 0 \end{bmatrix} = \frac{1}{R_1 + R\,\bar{s}_1^2} \begin{bmatrix} V_g \\ \bar{s}_1\, R\, V_g \end{bmatrix}$$

If $V_g = 100$ V, $\bar{s}_1 = 0.5$, $R_1 = 5$ Ohm and $R = 100$ Ohm, (see files "`Boost_DC_R100.m`" and "`Boost_DC_R100_mdl.slx`") the equilibrium point is:

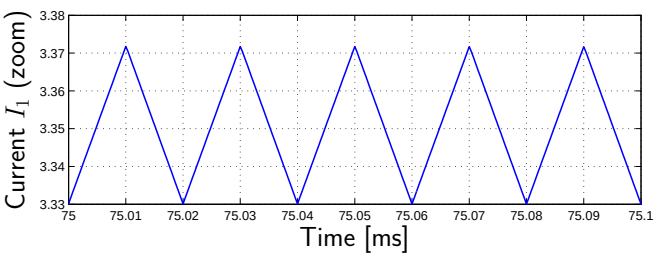
$$\begin{bmatrix} I_1 \\ V_0 \end{bmatrix} = \begin{bmatrix} 3.33 \text{ A} \\ 166.66 \text{ V} \end{bmatrix}$$

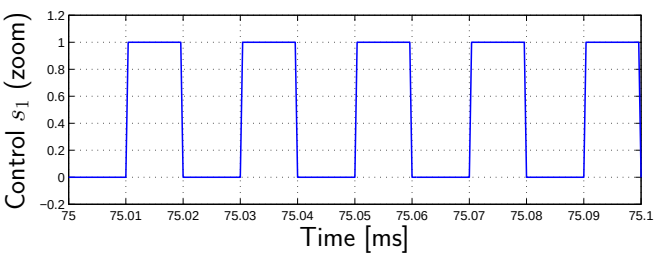
- State space trajectory starting from the origin, current I_1 and voltage V_0:





- Current I_1 (zoom) and control variable s_1 (zoom):





- PWM switching period: $T_c = 20\mu s$.

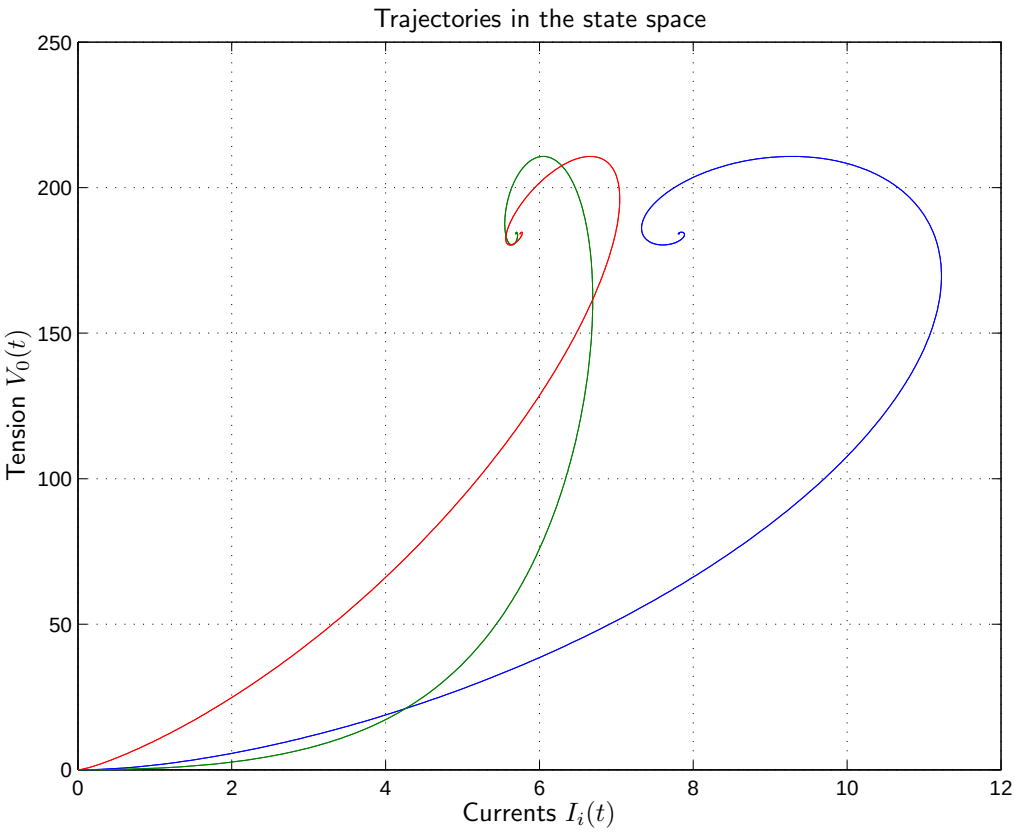
Case $m = 3$. For the simulation of an Interleaved Boost Converter of order m with all independent parameters please refer to files "`Boost_DC_M.m`" and "`Boost_DC_M_mdl.slx`". The following parameters have been used: number of inductors $m = 3$; input voltage $V_g = 100$ V; main capacitor $C = 200$ μF; load resistance $R = 20$ Ohm; self inductance coefficients $[L_1,\ L_2,\ L_3] = [20,\ 18,\ 22]$ mH; resistances of the inductors $[R_1,\ R_2,\ R_3] = [1,\ 3,\ 2]$ Ohm; duty cycles $[\bar{s}_1,\ \bar{s}_2,\ \bar{s}_3] = [0.5,\ 0.45,\ 0.48]$. The mutual inductance coefficients M_{ji} between inductors L_i and L_j have been defied as follows:

$$M_{ji} = M_c \sqrt{L_i\, L_j}$$

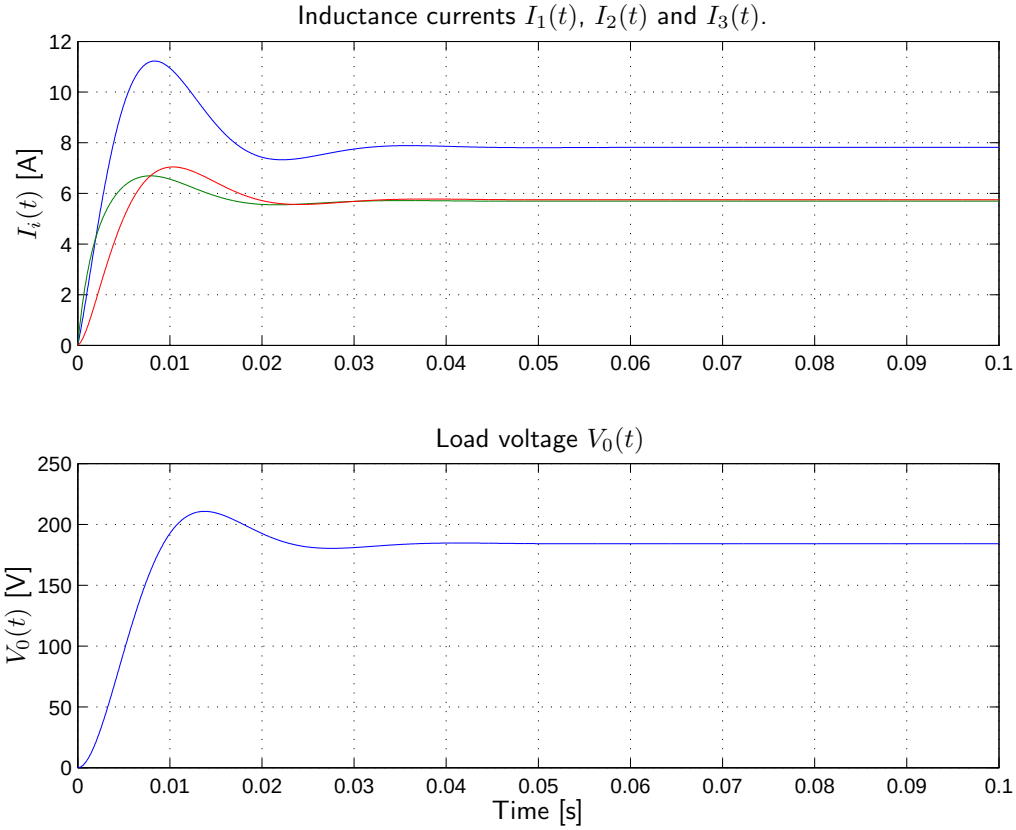
where $0 < M_c < 1$. For the m-order case, the equilibrium point can be computed as follows:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \hline V_0 \end{bmatrix} = - \left[\begin{array}{ccc|c} -R_1 & 0 & 0 & -s_1 \\ 0 & -R_2 & 0 & -s_2 \\ 0 & 0 & -R_3 & -s_3 \\ \hline s_1 & s_2 & s_3 & -\frac{1}{R} \end{array}\right]^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \hline 0 \end{bmatrix} [V_g] = \begin{bmatrix} 7.81 \text{ A} \\ 5.67 \text{ A} \\ 5.75 \text{ A} \\ 184.38 \text{ V} \end{bmatrix}$$

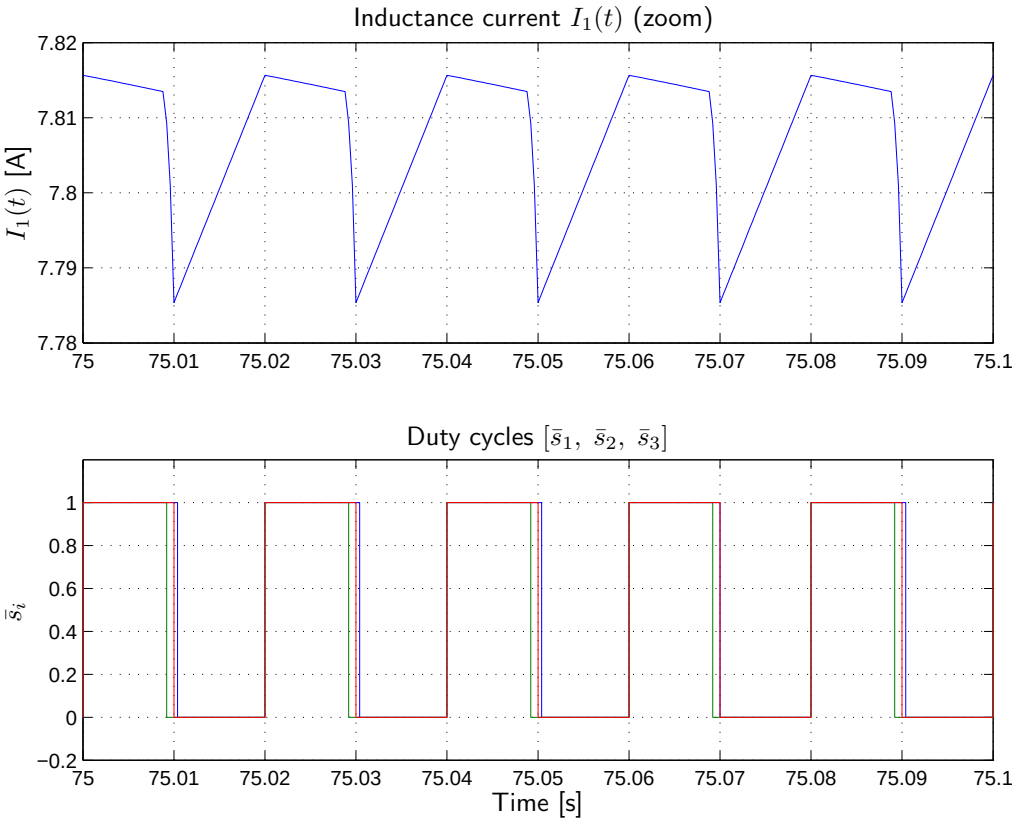
- State space trajectories for the considered 3-th order case.



- Inductance currents $I_1(t)$, $I_2(t)$, $I_3(t)$ and load voltage V_0.



- Inductance currents $I_1(t)$, $I_2(t)$, $I_3(t)$ and load voltage V_0 (zoom).



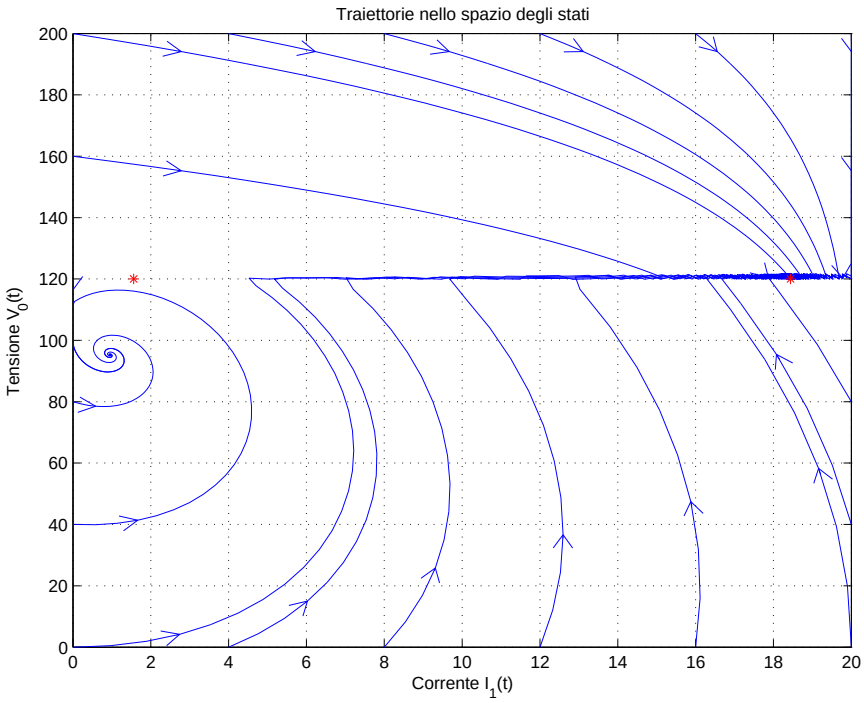
- PWM switching period: $T_c = 20\mu s$.

2) Feedback control

Let V_{ref} denote the desired output voltage, let $\lambda(V_0) = 0$ be the sliding surface and let us consider the following control law $s_1(V_0)$:

$$\lambda(V_0) = V_{ref} - V_0 \qquad \Rightarrow \qquad s_1(V_0) = \frac{1 + \text{sign}(\lambda(V_0))}{2} = \begin{cases} 1 & \text{if } \lambda(V_0) \geq 0 \\ 0 & \text{if } \lambda(V_0) < 0 \end{cases}$$

The obtained trajectories in the state space $(I_1,\ V_0)$ (see files "`Boost_DC_Sliding.m`" and "`Boost_DC_Sliding_mdl.mdl`") are the following:



The dynamics of the system on the sliding surface $\lambda(V_0) = V_{ref} - V_0 = 0$ can be obtained imposing $\dot{\lambda}(V_0) = 0$:

$$\dot{V}_0 = 0 \qquad \rightarrow \qquad s_1 I_1 - \frac{1}{R} V_{ref} = 0 \qquad \rightarrow \qquad s_1 = \frac{V_{ref}}{R\, I_1} = \bar{s}_1$$

where $\bar{s}_1$ is the equivalent control. The control variable $s_1 \in \{0,\ 1\}$ theoretically switches at infinite frequency with duty-cycle equal to $\bar{s}_1 \in [0,\ 1]$. The duty-cycle cannot exceed the maximum value:

$$\bar{s}_1 < 1 \qquad \rightarrow \qquad \boxed{I_1 > \frac{V_{ref}}{R}}$$

When parameters R and I_1 change, the duty-cycle $\bar{s}_1$ changes such to guarantee that $V_0 = V_{ref}$. Substituting $\bar{s}_1$ in the first equation of system (2) one

obtains the following nonlinear differential equation:

$$\dot{I}_1 = -\frac{R_1 I_1}{L_1} - \frac{V_{ref}^2}{R\,L_1\,I_1} + \frac{V_g}{L_1} \qquad \leftrightarrow \qquad \dot{I}_1 = f(I_1) \tag{4}$$

The equilibrium points $I^*_{1a,1b}$ of this equation are obtained when $\dot{I}_1 = 0$:

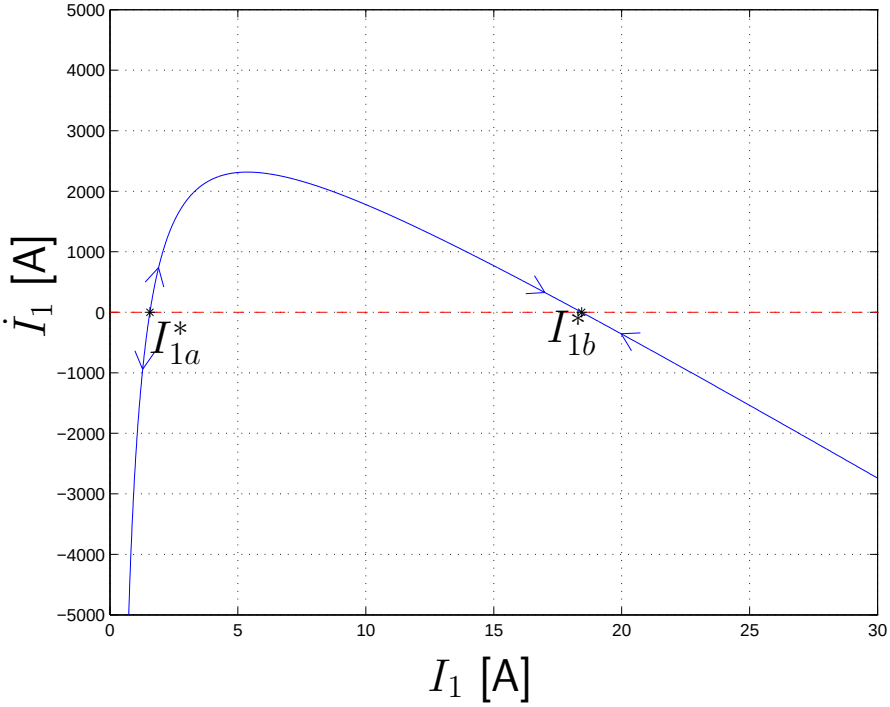
$$R_1 R\, I_1^2 - V_g R\, I_1 + V_{ref}^2 = 0 \qquad \rightarrow \qquad I^*_{1a,1b} = \frac{V_g R \mp \sqrt{V_g^2 R^2 - 4\,R_1 R V_{ref}^2}}{2\,R_1 R}$$

The equilibrium points $I^*_{1a,1b}$ are real only if:

$$V_g^2 R^2 - 4\,R_1 R V_{ref}^2 > 0 \qquad \rightarrow \qquad \boxed{V_{ref} < \sqrt{\frac{R}{4R_1}}\,V_g}$$

Qualitative behavior of variable $\dot{I}_1$ as a function of current I_1, see eq. (4).

Function $\dot{I}_1 = \dot{I}_1(I_1)$



From this figure it is evident that the equilibrium point I^*_{1a} is unstable, while the equilibrium point I^*_{1b} is stable. Equation (4) can be linearized computing the Jacobian of function $f(I_1)$:

$$J(I_1) = \frac{\partial f(I_1)}{\partial I_1} = -\frac{R_1}{L_1} + \frac{V_{ref}^2}{R\,L_1\,I_1^2}$$

One can easily verify that $J(I^*_{1a}) > 0$ and $J(I^*_{1b}) < 0$, that is point I^*_{1a} is unstable and point I^*_{1b} is stable.