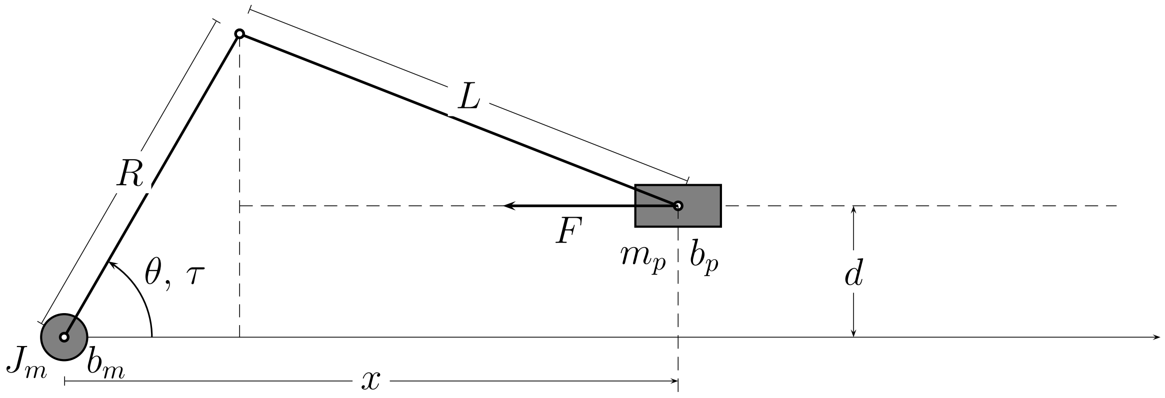
The crank-connecting rod system

The POG dynamic model

Let us consider the following crank-connecting rod system:



The position $x(\theta)$ of the piston can be expressed asfollows:

$$x(\theta) = R\cos\theta + \sqrt{L^2 - (R\sin\theta - d)^2}$$

Velocity $\dot{x}$ is obtained as follows:

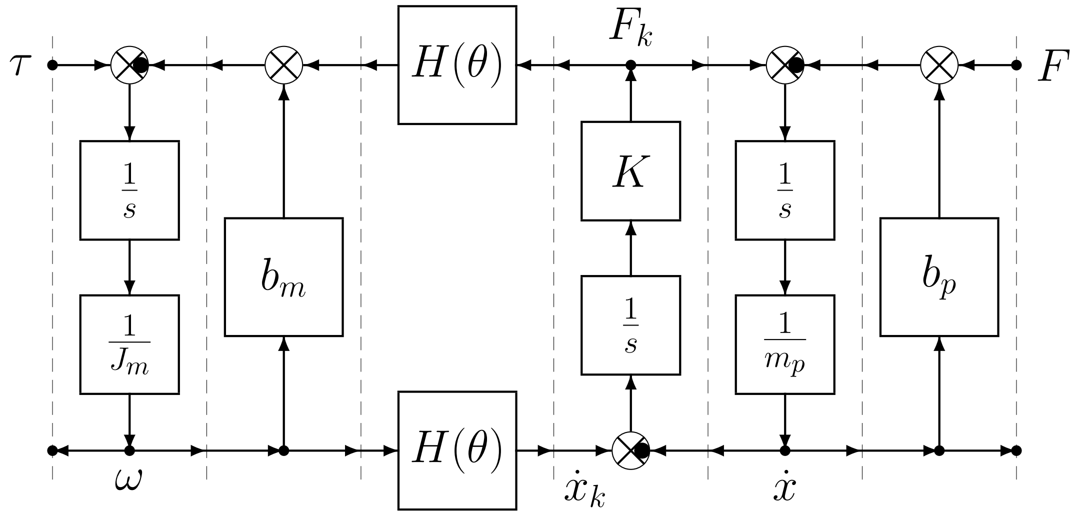
$$\begin{aligned}\dot{x}(t) &= R\left[-\sin\theta - \frac{(R\sin\theta - d)\cos\theta}{\sqrt{L^2 - (R\sin\theta - d)^2}}\right]\dot{\theta}\\ &= \underbrace{R\left[-\sin\theta - \frac{(\sin\theta - \beta)\cos\theta}{\sqrt{\alpha^2 - (\sin\theta - \beta)^2}}\right]}_{H(\theta)}\omega \;=\; H(\theta)\,\omega\end{aligned}$$

Function $H(\theta)$ and parameters α and β are defined as follows:

$$H(\theta) = \frac{\partial x(\theta)}{\partial \theta}, \qquad \alpha = \frac{L}{R} > 1, \qquad \beta = \frac{d}{R} < 1.$$

The time-varying dynamic model of the considered rigid system can be obtained adding a stiffness element K between the connecting rod and the piston and then let $K \to \infty$.

The POG dynamic model of the extended dynamic system with the additional stiffness element K is:



The POG state space model of the considered system is:

$$\underbrace{\begin{bmatrix} J_m & & \\ & K^{-1} & \\ & & m_p \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \dot{\omega} \\ \dot{F}_k \\ \ddot{x} \end{bmatrix}}_{\dot{\mathbf{x}}} = - \underbrace{\begin{bmatrix} b_m & H(\theta) & 0 \\ -H(\theta) & 0 & 1 \\ 0 & -1 & b_p \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \omega \\ F_k \\ \dot{x} \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \tau \\ F \end{bmatrix}}_{\mathbf{u}}$$

that is $\mathbf{L}\,\dot{\mathbf{x}} = -\mathbf{A}\,\mathbf{x} + \mathbf{B}\mathbf{u}$. When the fictitious stiffness $K \to \infty$ goes to infinity the state space variables are constrained as follows

$$\dot{x} = H(\theta)\,\omega.$$

Applying the following congruent state space transformation

$$\mathbf{x} = \mathbf{T}\,\omega, \qquad \Leftrightarrow \qquad \underbrace{\begin{bmatrix} \omega \\ F_k \\ \dot{x} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ H(\theta) \end{bmatrix}}_{\mathbf{T}} \omega$$

one obtains the following transformed and reduced system:

$$\frac{d[J(\theta)\,\omega]}{dt} - N_1(\theta)\omega = -b(\theta)\,\omega + \overline{\mathbf{B}}(\theta)\,\mathbf{u} \tag{1}$$

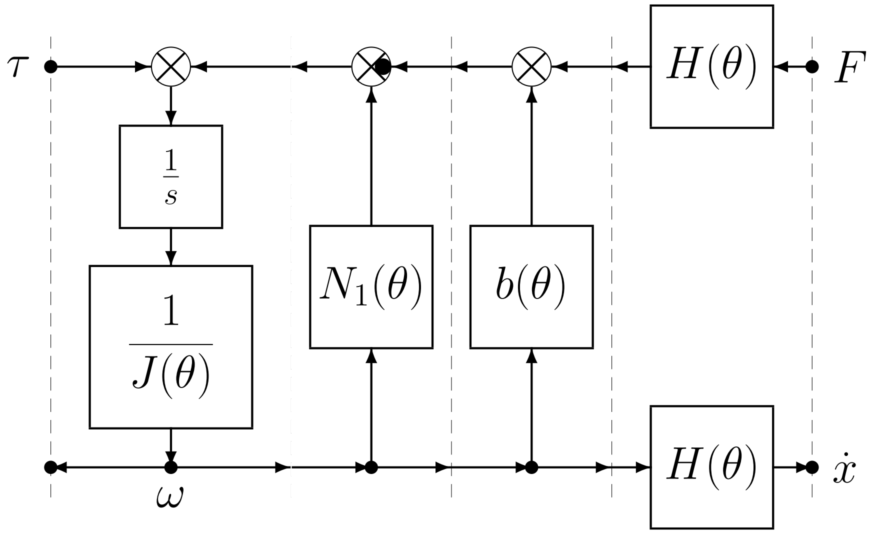
where:

$$J(\theta) = \mathbf{T}^T\mathbf{L}\mathbf{T} = J_m + H^2(\theta)\,m_p, \quad N_1(\theta) = \dot{\mathbf{T}}^T\mathbf{L}\mathbf{T} = \frac{\dot{J}(\theta)}{2} = m_p\,H(\theta)\dot{H}(\theta)$$

and

$$b(\theta) = \mathbf{T}^T\mathbf{A}\mathbf{T} = b_m + H^2(\theta)\,b_p, \qquad \overline{\mathbf{B}}(\theta) = \mathbf{T}^T\mathbf{B} = \begin{bmatrix} 1 & -H(\theta) \end{bmatrix}.$$

POG graphical representations of the reduced system:



The dynamic equation (1) can also be written as follows:

$$\frac{d}{dt}(J(\theta)\,\omega) - \frac{\dot{J}(\theta)}{2}\,\omega = -(b_m + H^2(\theta)\,b_p)\,\omega + \tau - H(\theta)F \qquad (2)$$

The Lagrange approach

Lagrange Equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \qquad i = 1, \ldots, N$$

where

- N degrees of freedom of the mechanical system;
- q_i generalized Lagrangian coordinates;
- Q_i generalized forces;
- T kinetic energy of the system (masses and inertias);
- U potential energy of the system (springs and gravitational forces);

The kinetic energy of the crank-connecting rod system is the following:

$$T = \frac{1}{2}J_m\dot{\theta}^2 + \frac{1}{2}m_p\dot{x}^2 = \frac{1}{2}J_m\dot{\theta}^2 + \frac{1}{2}m_p\left[H(\theta)\,\dot{\theta}\right]^2$$

The potential energy is zero: $U = 0$. In this case the Lagrangian coordinate is $q_1 = \theta$. Intermediate terms of the Lagrangian Equations:

$$\begin{aligned}
\frac{\partial T}{\partial \dot\theta} &= J_m \dot\theta + m_p H^2(\theta)\,\dot\theta \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) &= J_m \ddot\theta + m_p H^2(\theta)\,\ddot\theta + 2\,m_p H(\theta)\,\dot H(\theta)\,\dot\theta \\
\frac{\partial T}{\partial \theta} &= m_p H(\theta)\,\dot\theta\,\frac{\partial H(\theta)}{\partial \theta}\,\dot\theta = m_p H(\theta)\,\dot H(\theta)\,\dot\theta
\end{aligned}$$

The external generalized force Q is obtained as follows:

$$Q\,d\theta = \tau\,d\theta - b_m \dot\theta\,d\theta - (b_p\,\dot x + F)\,dx = \tau\,d\theta - b_m\dot\theta\,d\theta - (b_p\,H(\theta)\dot\theta + F) H(\theta)\,d\theta$$

from which one obtains

$$Q = \tau - b_m\dot\theta - b_p\,H^2(\theta)\dot\theta - H(\theta)\,F.$$

The Lagrangian equation of the system can be written as follows:

$$\frac{d}{dt}\Big(J_m\dot\theta + m_p H^2(\theta)\,\dot\theta\Big) - m_p H(\theta)\,\dot H(\theta)\,\dot\theta = \tau - b_m\dot\theta - b_p\,H^2(\theta)\dot\theta - H(\theta)\,F. \tag{3}$$

Using the following correspondences:

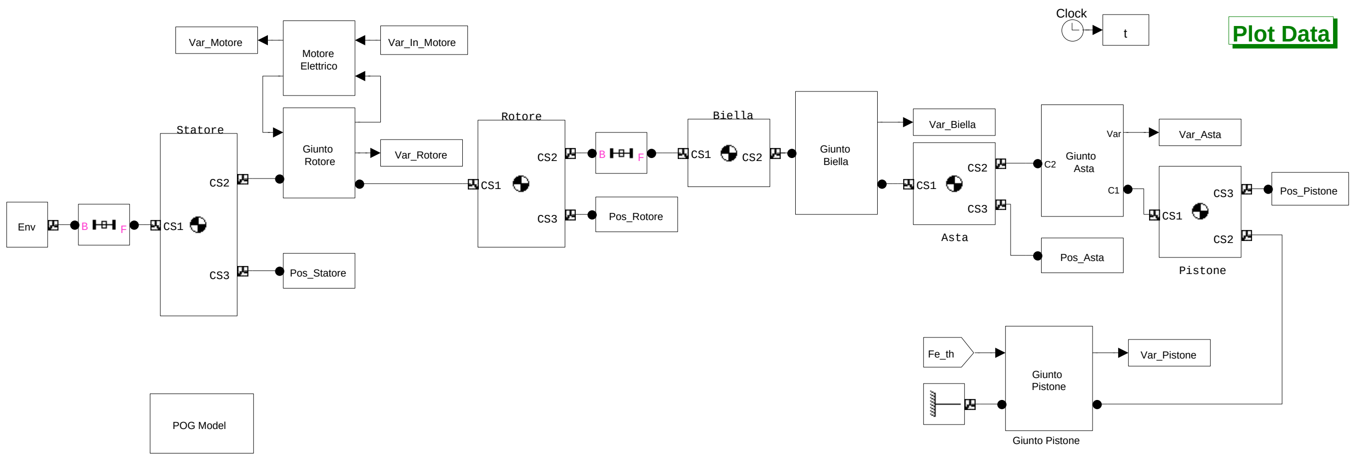
$$\frac{\partial T}{\partial \dot\theta} = J(\theta)\,\dot\theta, \qquad \frac{\partial T}{\partial \theta} = N_1(\theta)\,\dot\theta = N_2(\theta)\,\dot\theta, \qquad b(\theta) = b_m + b_p\,H^2(\theta)$$

equation (3) is equivalent to equation (1).

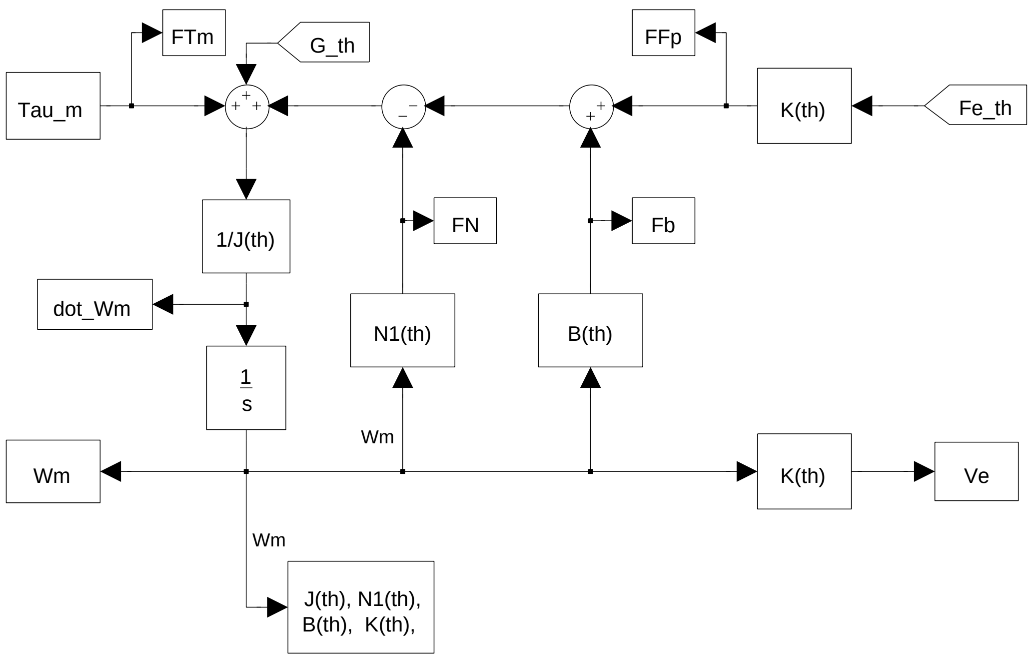
Simulation of the crank and connecting-rod system

The Matlab/Simulink block schemes

The crank and connecting-rod system has been simulated in Matlab/Simulink. The SimMechanic block scheme of the crank and connecting-rod system:

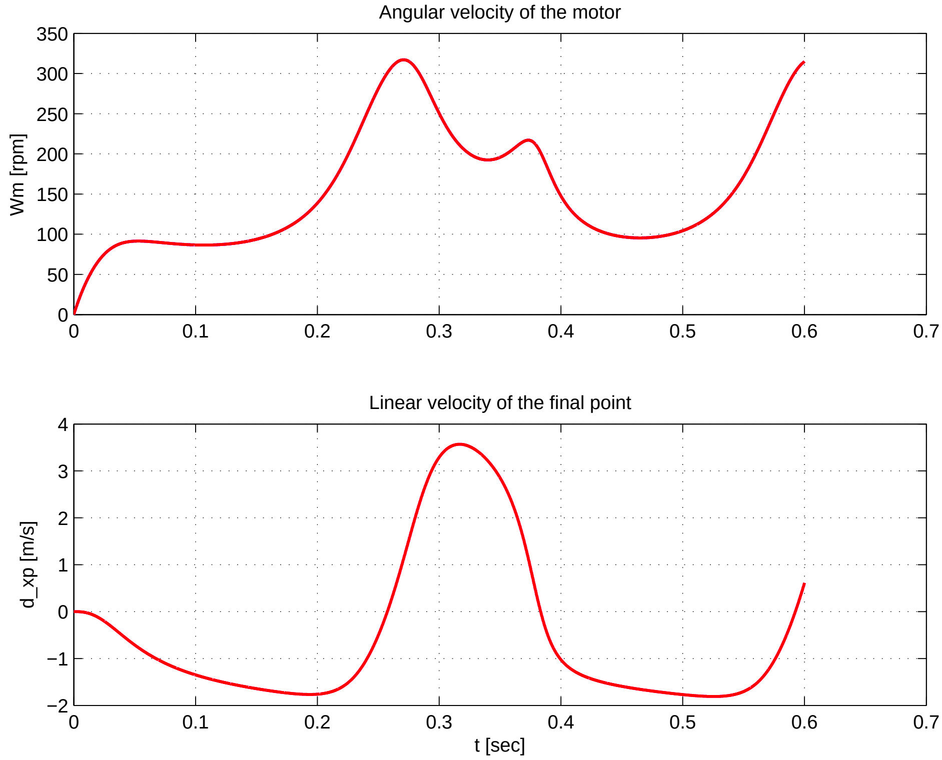


POG block scheme of the crank and connecting-rod system:



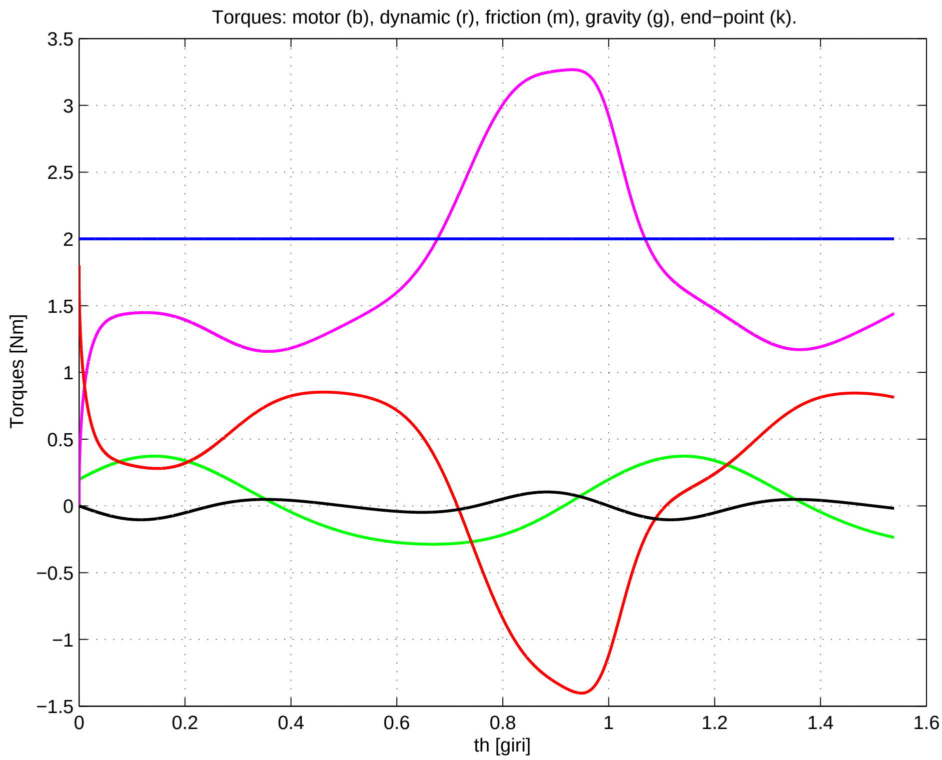
The two systems have been simulated considering a constant input torque: $\tau = 2$ Nm.

Angular velocity of the motor ω and linear velocity of the final point V_e :

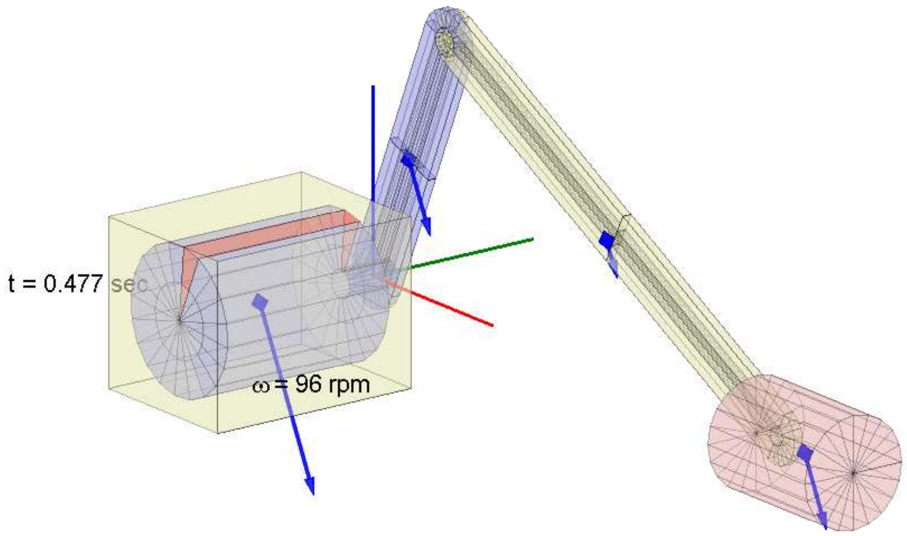


The two block schemes provide the same results. Maximum error: 0.00071273.

Time behavior of the system torques: the motor torque (blue), the friction torque (magenta), the dynamic torque (red), the gravity force (—green green) and the final-point torque (black):

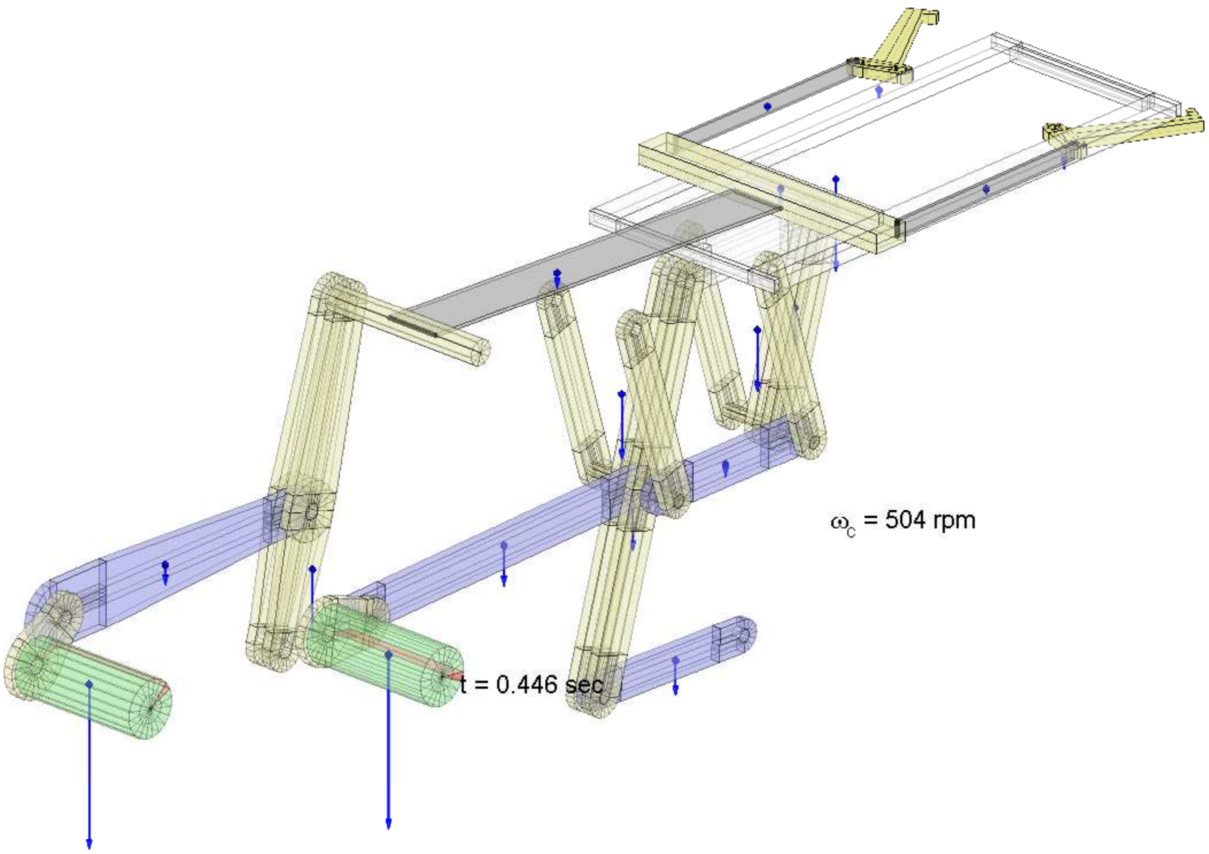


A three-dimensional graphical representation of the crank and connecting-rod system:

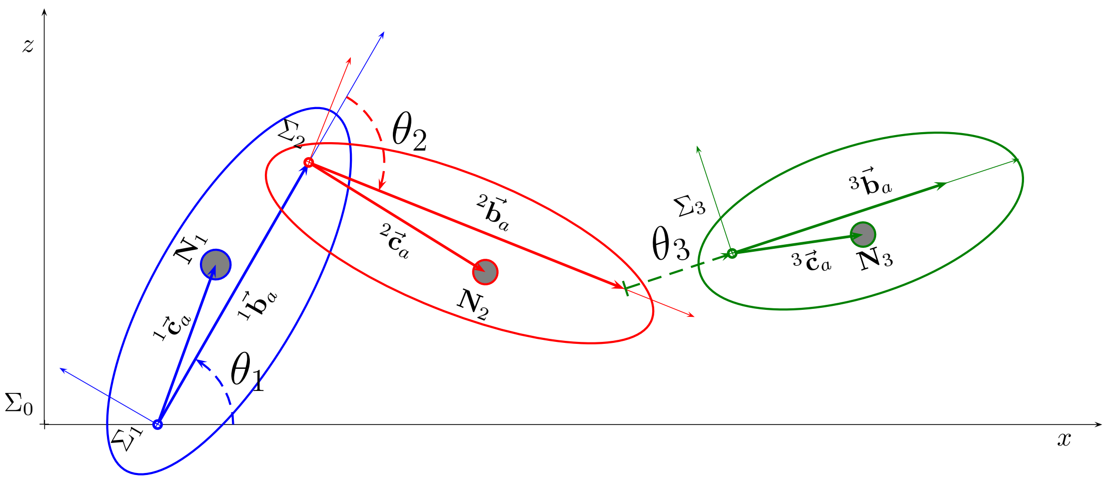


The approach can be extended to more complex systems.

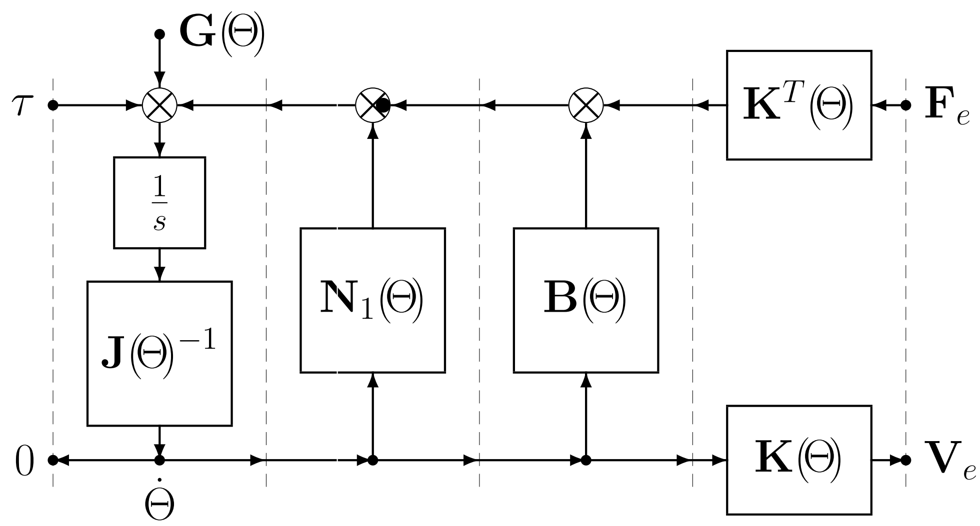
Example: the Trolley-Folder system:



The approach can be applied also to robotic systems.



POG block scheme of the system:



Dynamic equations of the system:

$$\frac{d[\mathbf{J}(\Theta)\,\dot{\Theta}]}{dt} - \mathbf{N}_1(\Theta) = -\mathbf{B}(\Theta)\,\dot{\Theta} + \mathbf{G}(\Theta) - \mathbf{K}^T(\Theta)\mathbf{F}_e + \tau$$

$\mathbf{J}(\Theta)$ is the inertia matrix, $\mathbf{N}_2(\Theta)$ is the dynamical matrix, $\mathbf{B}(\Theta)$ is the friction matrix, $\mathbf{H}(\Theta)$ is the Jacobian matrix, $\mathbf{K}(\Theta)$ is the final point matrix, $\mathbf{G}(\Theta)$ is the gravity vector, $\mathbf{F}_e$ is the final point force vector, $\mathbf{V}_e$ is the final point velocity vector and τ is the input torque vector.